Quantum Combinatorics

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Lake Michigan Workshop on
Combinatorics & Graph Theory

Purdue.

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Let's count [~ 50,000 BC*]

$\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, the symmetric group on an n elt. set.

$$\sum_{\pi \in \mathfrak{S}_n} 1 = n! = n(n-1) \cdots 2 \cdot 1.$$

$$\sum_{\substack{S \subseteq \{1, \dots, n\} \\ |S| = k}} 1 = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$= \frac{n(n-1) \cdots (n-k+1)}{k!}$$

* Source: Wikipedia

Let's q-count [1700's Euler*]

q-analogue of $n \in \mathbb{Z}^+$

$$[n]_q = [n] = 1 + q + \cdots + q^{n-1},$$

q an indeterminate.

$$\lim_{q \to 1} [n]_q = \underbrace{1 + \cdots + 1}_{n} = n.$$

$$[n]! = [n][n-1] \cdots [2] \cdot [1]$$

* Theta functions

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\binom{n+1}{2}} b^{\binom{n}{2}},$$

$$|ab| < 1.$$

[Netto 1901]

inversion statistic on permutations

(see also Cramer 1750, Laplace 1772, Bézout 1764).

$$\mathrm{inv}(\pi) = \left| \{ (i,j) : \pi_i > \pi_j \text{ for } i<j \} \right|.$$

ex.

π	inv π
123	0
132	1
213	1
231	2
312	2
321	3

Theorem: [MacMahon 1916].

$$\sum_{\pi \in \mathfrak{S}_n} q^{\mathrm{inv}(\pi)} = [n]_q !$$

~~This is~~ a combinatorial interpretation of $[n]_q !$

ex.

π	$q^{\operatorname{inv}\pi}$
123	q^0
132	q^1
213	q^1
231	q^2
312	q^2
321	q^3

$$\sum = 1 + 2q + 2q^2 + q^3$$
$$= 1 \cdot (1+q)(1+q+q^2)$$
$$= [3]_q \, !$$

def. The ~~Gaussian~~ polynomial or q-binomial

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}$$

$$\left(= \frac{[n]_q\,[n-1]_q \cdots [n-k+1]_q}{[k]_q!} \right).$$

ex. $\mathfrak{S}(0^{n-k}, 1^{k})$.

$n=4, \quad k=2.$

π	$q^{\mathrm{inv}(\pi)}$
0011	q^0
0101	q^1
0110	q^2
1001	q^2
1010	q^3
1100	q^4

$$\frac{[4]_q\,[3]_q}{[2]_q} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q.$$

$$\parallel$$

$$\sum = q^4 + q^3 + 2q^2 + q + 1 = (1+q^2)(1+q+q^2) \cdot \frac{(1+q)}{(1+q)}$$

Theorem: [MacMahon 1916].

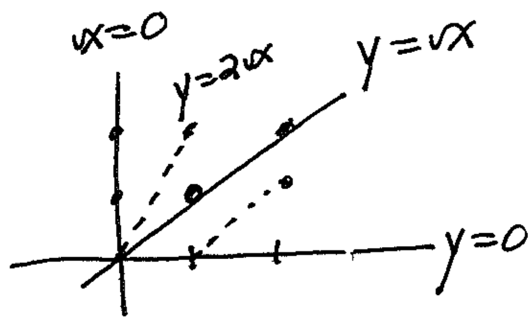
$$\sum_{\pi \in \mathfrak{S}(0^{n-k},\, 1^{k})} q^{\operatorname{inv} \pi} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

Other combinatorial interpretations

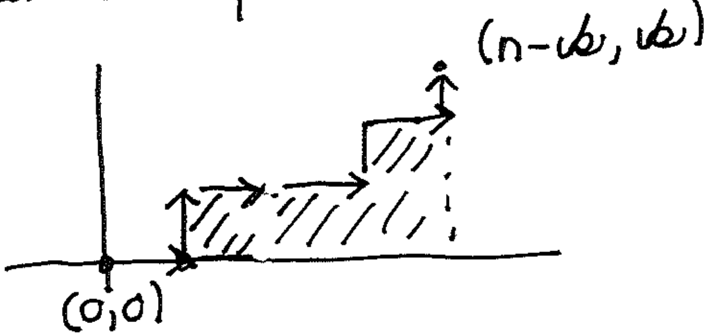
① $\begin{bmatrix} n \\ k \end{bmatrix} =$ # k-dim'l subspaces of an n-dim'l v.s. over $\mathbb{F}_q$.

ex. $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1+q$.

When $q=3$:



② lattice paths using $n-k$ E's + k N's, weighted by area under path.



Unimodality

A sequence $\{a_0, a_1, \dots, a_m\}$ is **unimodal** if

$$a_0 \le a_1 \le \dots \le a_j \ge \dots \ge a_m$$

for some $0 \le j \le m$.

ex. row of Pascal's triangle

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}$$

Unimodality of $\begin{bmatrix} n \\ k \end{bmatrix}$

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{(n-k)\cdot k} + \cdots + a_1 q^1 + a_0.$$

Theorem: [Sylvester 1878].
The coeffs of $\begin{bmatrix} n \\ k \end{bmatrix}$ are unimodal, i.e.,

$$a_0 \le \cdots \le a_j \ge \cdots \ge a_{(n-k)k}.$$

[O'Hara 1990] Gave the first combinatorial proof of this result.

O'Hara's proof

Fix n and k.

Weight a k-subset of $\{1, \ldots, n\}$ by

$$wt(S) = \sum_{i=1}^{k} s_i.$$

Form a poset = partially ordered set.

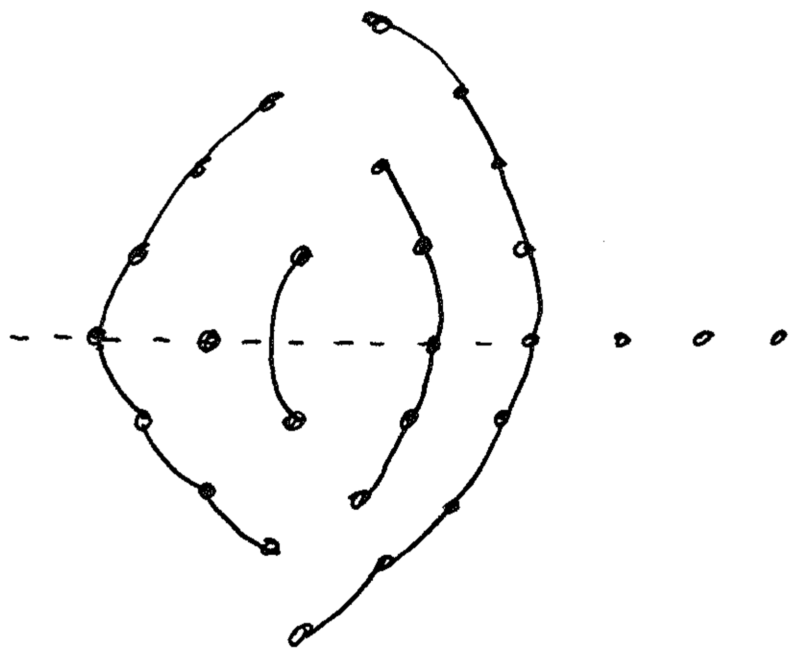
ex. $\begin{bmatrix}4\\2\end{bmatrix} = q^4 + q^3 + 2q^2 + q + 1.$

$$\begin{tikzcd}[column sep=small]
& 3,4 \arrow[d, no head] & \\
& 2,4 \arrow[dl, no head] \arrow[dr, no head] & \\
1,4 \arrow[dr, no head] & & 2,3 \arrow[dl, no head] \\
& 1,3 \arrow[d, no head] & \\
& 1,2. &
\end{tikzcd}$$

	wt	wt − 3
3,4	7	4
2,4	6	3
1,4 2,3	5	2
1,3	4	1
1,2.	3	0.

Construct a symmetric chain decomposition (SCD):

(Write P as a disjoint union of rank-symmetric saturated chains).



$\Rightarrow$ unimodal.

Theorem: [Pak - Panova, 2013]

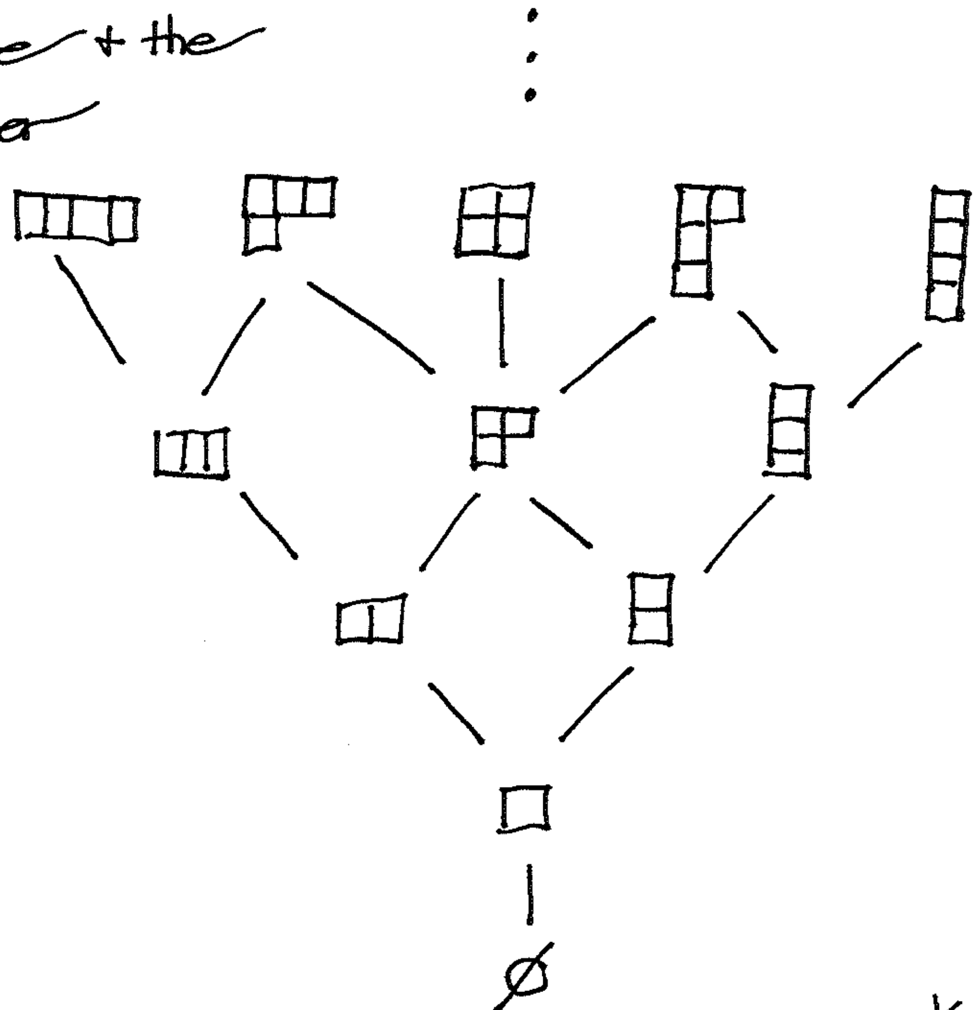
For $j, k \geq 8$ in $\begin{bmatrix} j+k \\ k \end{bmatrix}$, the coeffs satisfy.

$$a_1 < \cdots < a_{\lfloor jk/2 \rfloor} = a_{\lceil jk/2 \rceil} > \cdots > a_{jk-1}.$$

Proof: Algebraic.

Uses combinatorics of Young tableau, + semigroup property of Kronecker coeffs of $\mathfrak{S}_n$ representations.

Young's lattice + the poset of integer partitions.



L(m,n) = partitions whose Ferrers diagram fits inside an m×n rectangle.

rank generating function of L(m,n) $= \begin{bmatrix} m+n \\ m \end{bmatrix}$

Open: Find a SCD for L(m,n).

[Stanton 1990].

F_λ not unimodal for $\lambda = (8,8,4,4)$:

$$1, 1, 2, 3, 5, 6, 9, 11, 15, 17, 21, 23, 27, 28, \underline{31, 30, 31}, 27, 24, 18, 14, 8, 5, 2, 1$$

Open: Classify non-unimodal partitions.

[Stanley-Zanello 2015].

F_λ unimodal for the shifted Ferrers diagram $\lambda = \langle n, n-1, n-2, n-3 \rangle$, $n \geq 4$.

Conjecture: [Stanley-Zanello].

F_λ unimodal from shifted Ferrers diagrams with "minimal" λ from arithmetic progressions.

Juggling & q-analogues

Assume: 1-handed juggler

Can catch and throw one ball

at a time.

Theorem: [Buhler - Eisenbud - Graham - Wright]

The # of juggling patterns of period d

and at most n balls is

$$n^d.$$

ex $n=2$, $d=3$

Throw vector

$(1,1,1)$

$(2,2,2)$

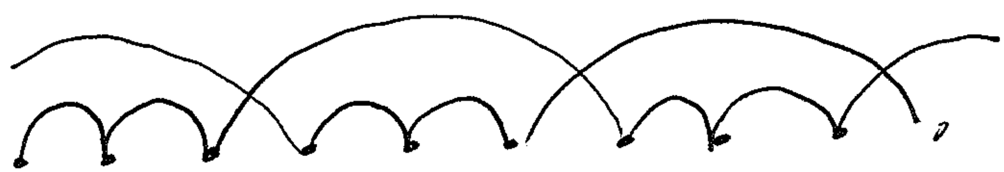
$(1,2,3)$

$(2,3,1)$

(shifted.

$(3,1,2)$

etc.



$(1,1,4)$

$(1,4,1)$

etc.

$(4,1,1)$.

etc.

$2^3=8$.

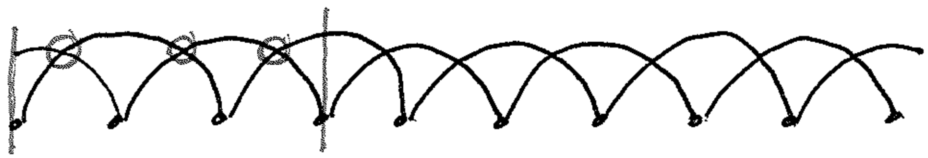
q. 20.

$q^{\# \text{crossings}}$

(1,1,1)

q^0

(2,2,2)

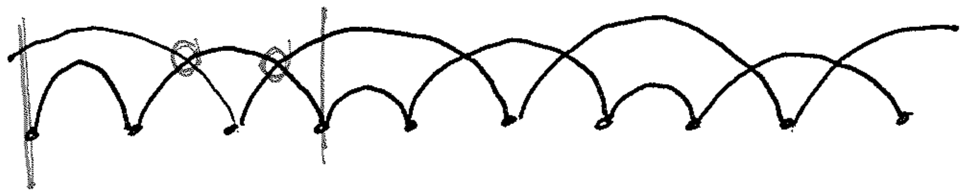


q^3

(1,2,3)

(2,3,1)

(3,1,2)

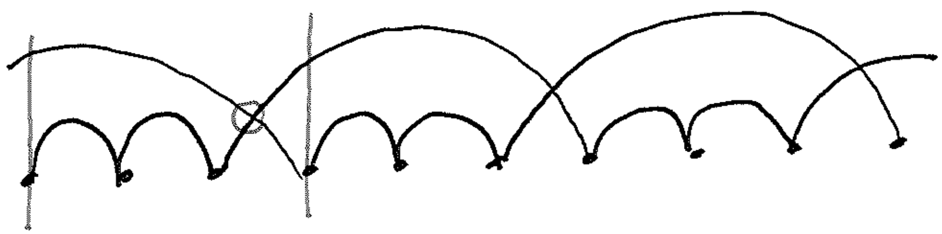


$3q^2$.

(1,1,4)

(1,4,1)

(4,1,1)



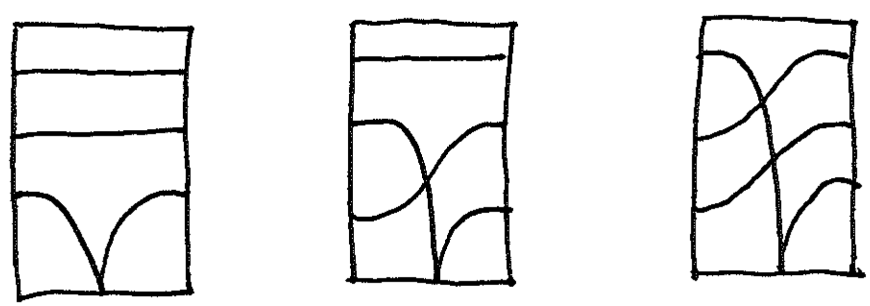
$3q^1$

$$(1+q)^3 = [2]^3.$$

Theorem: [Ehrenborg–Readdy].
The weight of juggling patterns of period d and at most n balls is

$$[n]^d.$$

Proof.
Cards for ≤ 3 balls.



q^0 $\qquad$ q^1 $\qquad$ q^2

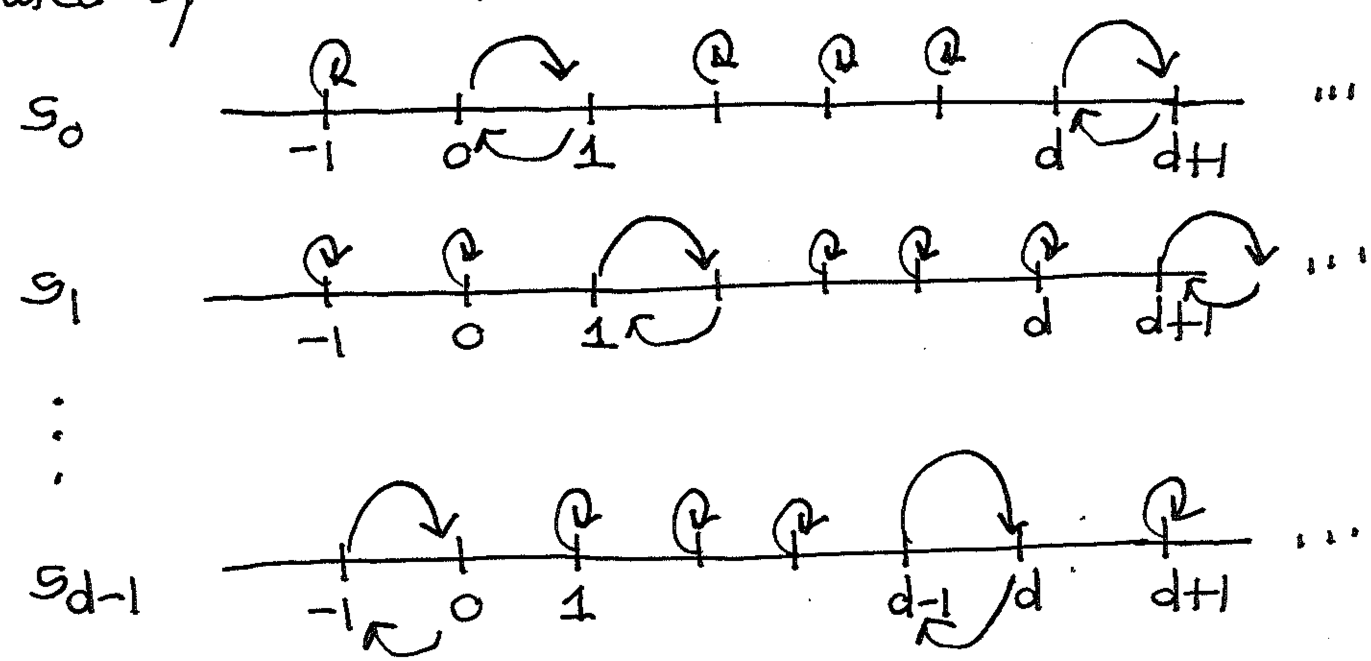
$\square$

Application: Affine Weyl group $\tilde{A}_{d-1}$.

def. [Lusztig]
$\tilde{A}_{d-1}$ is the group of bijections $\sigma : \mathbb{Z} \to \mathbb{Z}$ wrt composition satisfying

1. $\sigma(i+d) = \sigma(i) + d \quad \forall i$
2. $\displaystyle\sum_{i=1}^{d} (\sigma(i) - i) = 0$ "conservation of momentum"

Generated by the simple reflections.

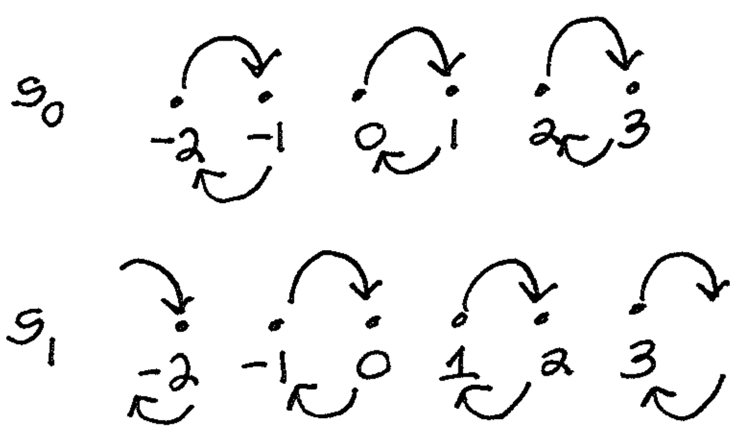


Theorem: [Bott]

The Poincaré ~~series~~ ~~for~~ $\tilde{A}_{d-1}$ is.

$$\sum_{\sigma \in \tilde{A}_{d-1}} q^{\ell(\sigma)} = \frac{1-q^d}{(1-q)^d}.$$

ex. $\tilde{A}_1$



$$\sum_{\sigma \in \tilde{A}_2} q^{\ell(\sigma)} = \frac{1-q^2}{(1-q)^2}$$

$$= \frac{1+q}{1-q}$$

$$= 1 + 2q + 2q^2 + 2q^3 + \cdots$$

A Combinatorial Proof [Ehrenborg-R].

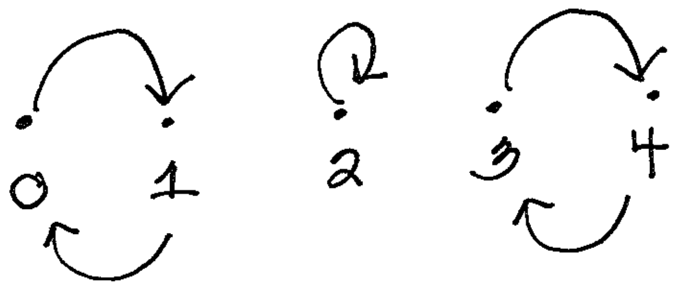
$$P_n = \{ \sigma \in \tilde{A}_{d-1} : \quad n > \max \{ i - \sigma(i) \} \}.$$

Add n to these $\sigma \in P_n$.

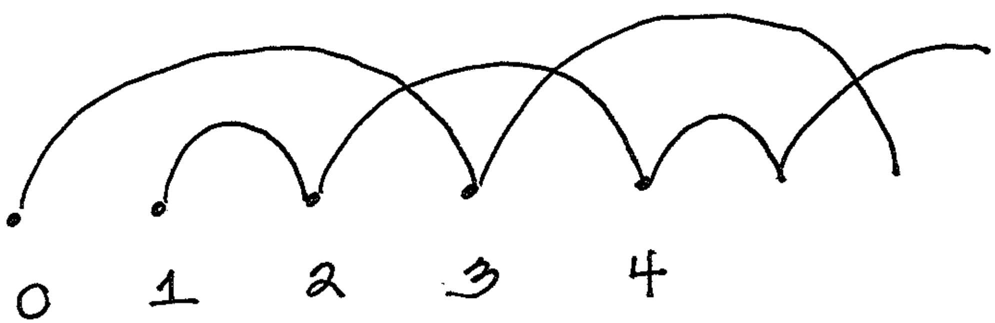
Claim: Are juggling sequences with.
$(n-1) \cdot d - \ell(\sigma)$ crossings,
period d + exactly n balls.

Pf.
Nontrivial $\boxtimes$

ex. $\tilde{A}_{3-1}$



Add $n=2$.



Ig $(3,1,2)$.

<u>Proof</u> (cont'd)

$$\sum_{\sigma \in P_n} q^{(n-1)d - \ell(\sigma)} = [n]^d - [n-1]^d.$$

$$\sum_{\sigma \in P_n} \left(\frac{1}{q}\right)^{\ell(\sigma)} = \frac{[n]^d - [n-1]^d}{q^{(n-1)d}}.$$

$$\sum_{\sigma \in P_n} q^{\ell(\sigma)} = q^{(n-1)d}\left(\left(1+\frac{1}{q}+\cdots+\frac{1}{q^{n-1}}\right)^d - \left(1+\frac{1}{q}+\cdots+\frac{1}{q^{n-2}}\right)^d \right).$$

$$= (q^{n-1}+q^{n-2}+\cdots+1)^d - (q^{n-1}+\cdots+q)^d$$

$$= [n]^d - (q\,[n-1])^d.$$

$$\sum_{\sigma \in P_n} q^{\ell(\sigma)} = \left(\frac{1-q^n}{1-q} \right)^d - q^d \left(\frac{1-q^{n-1}}{1-q} \right)^d$$

Now.

$$\bigcup_{n \geq 1} P_n = \tilde{A}_{d-1}$$

Let $n \to \infty$.

$$\sum_{\sigma \in \tilde{A}_{d-1}} q^{\ell(\sigma)} = \frac{1-q^d}{(1-q)^d}. \qquad \blacksquare$$

<u>Cyclic sieving phenomenon.</u> [Reiner - Stanton - White]

X finite set

C finite cyclic group acting on X

$f(q)$ = polynomial in q w/ nonneg. $\mathbb{Z}$-coeffs.

def. $(X, C, f(q))$ exhibits <u>CSP</u> if for all $g \in C$

$$|X^g| = f(\omega),$$ ω an nth root of unity, $n = |g|$.

where $X^g = \{x \in X : gx = x\}$.

ex. 2-subsets of $\{1,2,3,4\}$.

$$f(q) = \begin{bmatrix}4\\2\end{bmatrix} = 1 + q + 2q^2 + q^3 + q^4.$$

X: 12 13 14
23 24
34.

$g = (1,2,3)$, $|g| = 3$

ω a 3rd root of unity.

$$f(\omega) = 1 + \omega + 2\omega^2 + \omega^3 + \omega^4$$
$$= 2 + 2\omega + 2\omega^2$$
$$= 0 \Rightarrow \text{No fixed points.}$$

$g = (1,2)(3)$, $|g| = 2$.

$$f(-1) = 1 - 1 + 2 - 1 + 1$$
$$= 2.$$

Fixed points:

12 & 34.

$n \times n$ alternating sign matrices.

Entries are $0, \pm 1$.

Row + column sums are 1.

Nonzero entries alternate in sign.

ex. $n=3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Theorem: [Zeilberger 1996]

The number of $n \times n$ alternating sign matrices is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$$

[Stanton, ~ 2007].

The cyclic group of order 4 generated by rotation of $\pi/2$ on alternating sign matrices exhibits CSP with

$$X(q) = \prod_{k=0}^{n-1} \frac{[3k+1]!}{[n+k]!}$$

Open: ① No linear algebra proof.

② $X(q)$ is the generating function for descending plane partitions by weight.

$X(q)$ is not via a statistic on ASM's.

Thank yau!