Euler flag enumeration of Whitney stratified spaces

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P n-dim'l polytope

The <u>f-vector</u> $(f_0, \text{—}, f_{n-1})$

f_i = # i-dim'l faces

[Steinitz 1906] Characterized f-vectors of 3-dim'l polytopes

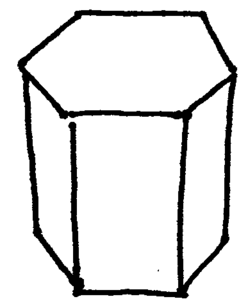
Open 20: Characterize f-vectors of n-dim'l polytopes, $n \geq 4$

[Stanley 1978; Billera-Lee 1980]
Done for simplicial polytopes

S	f_S	h_S	w_S
$\emptyset$	1	1	aaa
0	12	11	baa
1	18	17	aba
2	8	7	aab
01	36	7	bba
02	36	17	bab
12	36	11	abb
012	72.	1	bbb.

P, n-dim'l polytope

The <u>flag f-vector</u> f_S



The <u>flag h-vector</u>

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T.$$

[Stanley] $h_S = h_{\bar{S}}$.

The ab-index

$$\Psi(P) = \sum_{S} h_S \cdot w_S$$

ex. $\Psi($  $) = 1\,aaa + 11\,baa + 17\,aba + 7\,aab + 7\,bba + 17\,bab + 11\,abb + 1\,bbb$

$$= (a+b)^3 + 10\,(baa + aba + bab + abb) + 6\,(aba + aab + bba + bab)$$

$$= (a+b)^3 + 10\,(ab+ba)\,(a+b) + 6\,(a+b)\,(ab+ba)$$

$$= c^3 + 10\,dc + 6\,cd,$$

where $c = a+b$, $d = ab+ba$. The <u>cd-index</u>.

Theorem: [Bayer-Klapper 1991; Stanley 1994]
P polytope then $\Psi(P) \in \mathbb{Z}\langle c,d\rangle$.
P Eulerian poset then $\Psi(P) \in \mathbb{Z}\langle c,d\rangle$.

Eulerian: $\mu(x,y) = (-1)^{\rho(x,y)}$ for every interval $[x,y]$ in a graded poset P.

Equivalently,
in each non-trivial interval $[x,y]$:

elts of even rank = # elts of odd rank.

Some cd-history

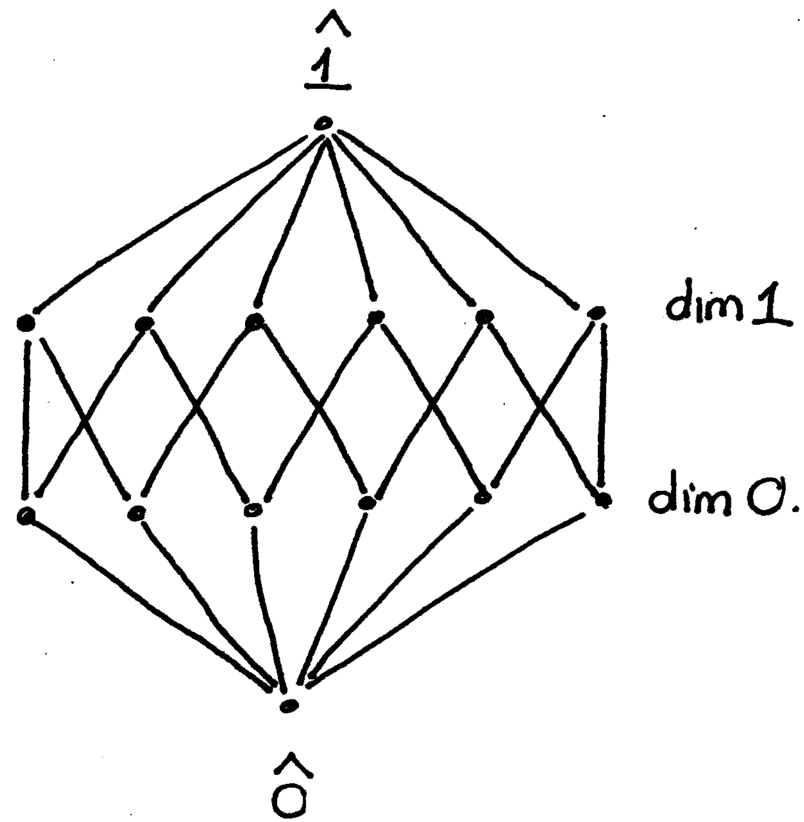
1980's	[Bayer-Billera]	Generalized Dehn-Sommerville relations.
1990's	[Bayer-Klapper].	Ψ removes all linear relations among flag vector entries
	[Stanley].	$\Psi \geq 0$ for $\mathcal{L}$(polytope), more generally, S-shellable face poset of regular CW-complex
	[Purtill]	n-simplex, n-cube $\Longleftrightarrow$ André + signed André permutations.
	[Ehrenborg-R]	Coalgebraic techniques
	[Billera-Ehrenborg-R]	Zonotopes span; OM / hyperplane arrangements
	[Billera-Ehrenborg]	Ψ(n-polytope) $\geq$ Ψ(n-simplex)

cd-history (cont'd)

2000's	[Karu]	Φ(Gorenstein* posets) ≥ 0
	[Karu-Ehrenborg]	Φ(Gorenstein* lattices) $\geq \Phi(B_n)$
	[Ehrenborg-R-Slone]	Arrangements of subtori
2010's	[Billera-Brenti]	Φ(Bruhat graphs) via quasi-symmetric fns; Kazhdan-Lusztig thy.
	[Ehrenborg-R]	Φ(Balanced graphs) via R-labelings.

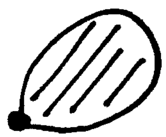
ex The n-gon $(n \geq 2)$.

S	f_S	h_S	w_S
$\emptyset$	1	1	aa
0	n	n−1	ba
1	n	n−1	ab
01	2n	1	bb.

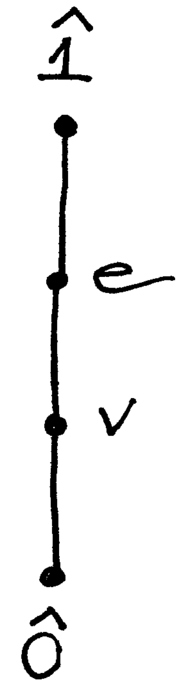


$\Phi($  $) = c^2 + (n-2)\,d.$

ex. 1-gon



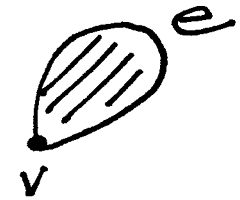
S	f_S	h_S
∅	1	1
0	1	0
1	1	0
01	1	0



Not Eulerian

Try again ...

S	$\overline{f}_S$	$\overline{h}_S = \sum_{T \subseteq S} (-1)^{\lvert S-T \rvert} \cdot \overline{f}_T$
$\emptyset$	1	1
0	1	0
1	1	0
01	2	1.



$\mathrm{link}_e(v) = \bullet \quad \bullet$

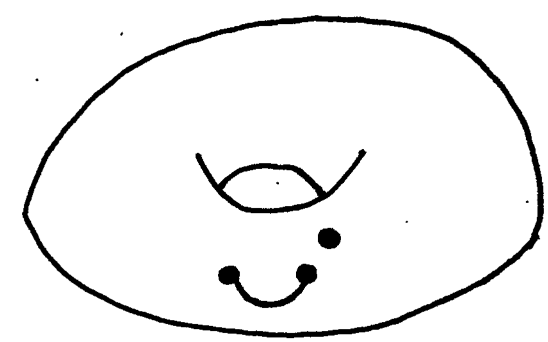
$\chi(\bullet \ \bullet) = 2,$

the Euler characteristic

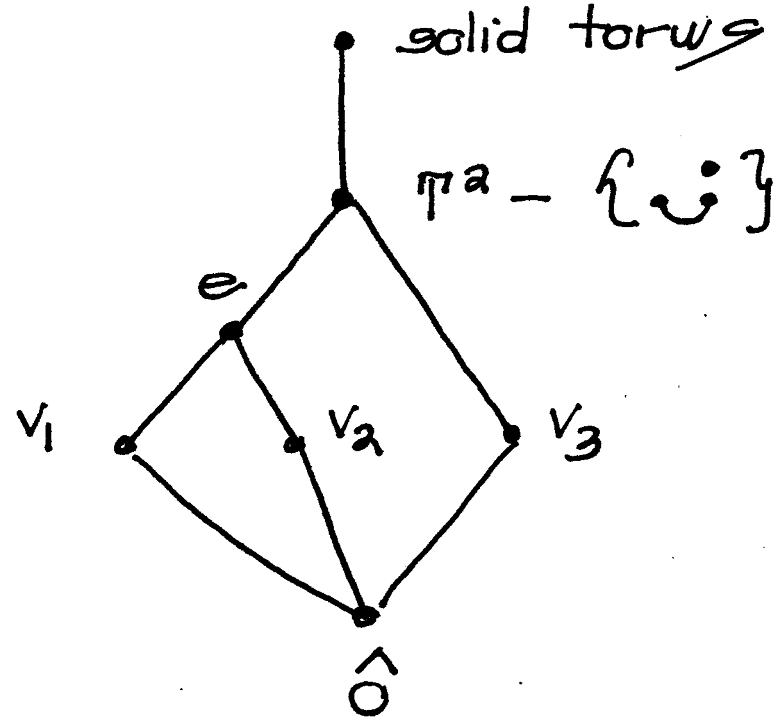
$\Phi($  $) = aa + bb$

$= c^2 - d.$

ex.



Face poset



Chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$
in the face poset weighted by.

$$\bar{\zeta}(c) = \chi(x_1)\cdot\chi(\mathrm{link}_{x_2}(x_1)) \cdots \chi(\mathrm{link}_{x_k}(x_{k-1}))$$

ex. (cont'd).

S	$\bar{f}_S$	$\bar{h}_S$	$3dc$	$-2cd$
$\emptyset$	0	0	0	0
0	3	3	3	0
1	1	1	3	−2
2	−2	−2	0	−2
01	2	−2	0	−2
02	2	1	3	−2
12	2	3	3	0
012	4	0	0	0

$\Phi($  $)$

$=$

$3dc - 2cd.$

These are examples of

<u>Whitney stratifications</u>

Subdivide space into strata:

$$W = \dot{\bigcup_{X \in P}} X$$

<u>Condition of the frontier</u>:

$X \cap \bar{Y} \neq \phi \iff X \subseteq \bar{Y} \iff X \leq_P Y$ in the face poset P.

Whitney conditions A + B:

No fractal behavior

No infinite wiggling ex. $x \cdot \sin\left(\frac{1}{x}\right)$

$\Rightarrow$ The links are well-defined.

Whitney stratifications exist for:

- real or complex algebraic sets
- analytic sets
- semi-analytic sets
- quotients of smooth manifolds by compact group actions.

The Fine Print

Definition Let W be a closed subset of a smooth manifold M, and suppose W can be written as a locally finite disjoint union

$$W = \bigcup_{X \in \mathcal{P}} X$$

where $\mathcal{P}$ is a poset. Furthermore, suppose each $X \in \mathcal{P}$ is a locally closed subset of W satisfying the *condition of the frontier*:

$$X \cap \overline{Y} \neq \emptyset \iff X \subseteq \overline{Y} \iff X \leq_{\mathcal{P}} Y.$$

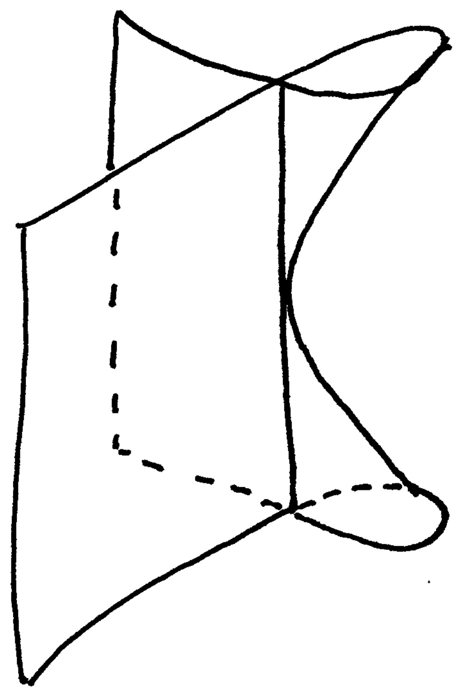
This implies the closure of each stratum is a union of strata. We say W is a *Whitney stratification* if

1. Each $X \in \mathcal{P}$ is a locally closed smooth submanifold of M (not necessarily connected).
2. If $X <_{\mathcal{P}} Y$ then Whitney's conditions (A) and (B) hold: Suppose $y_i \in Y$ is a sequence of points converging to some $x \in X$ and that $x_i \in X$ converges to x. Also assume that (with respect to some local coordinate system on the manifold M) the secant lines $\ell_i = \overline{x_i y_i}$ converge to some limiting line ℓ and the tangent planes $T_{y_i}Y$ converge to some limiting plane τ. Then the inclusions

   $$\text{(A) } T_x X \subseteq \tau \qquad \text{and} \qquad \text{(B) } \ell \subseteq \tau$$

   hold.

ex. The Whitney cusp.



Whitney stratifications (their face posets) are examples of ...

A <u>quasi-graded poset</u> $(P, \rho, \bar{\zeta})$ consists of

i. P finite poset with $\hat{0}$ + $\hat{1}$ (not necessarily graded)

ii. $\rho : P \to \mathbb{N}$ order-preserving ($x < y \Rightarrow \rho(x) < \rho(y)$)

iii. $\bar{\zeta} \in I(P)$, the weighted zeta function satisfying $\bar{\zeta}(x,x) = 1 \quad \forall x \in P$.

def. $(P, \rho, \bar{\xi})$ is <u>Eulerian</u> if.

$$\sum_{x \le y \le z} (-1)^{\rho(x,y)} \cdot \bar{\xi}(x,y) \cdot \bar{\xi}(y,z) = \delta_{x,z}.$$

Remark: $\bar{\xi} = \xi$ gives the <u>classical Eulerian condition</u>

$$\sum_{x \le y \le z} (-1)^{\rho(x,y)} = \delta_{x,z}.$$

Define

$$\Phi(P, \rho, \bar{\xi}) = \sum_{g} \bar{h}_g \cdot w_g$$

with

$$\bar{\xi}(c) = \bar{\xi}(x_0, x_1) \cdot \bar{\xi}(x_1, x_2) \cdots \bar{\xi}(x_{k-1}, x_k)$$

for a chain

$$c: \quad \hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}.$$

Theorem: $(P, \rho, \bar{\zeta})$ an Eulerian quasi-graded poset.
Then

$$\Psi(P, \rho, \bar{\zeta}) \in \mathbb{Z}\langle c,d \rangle.$$

Theorem: M manifold with a Whitney stratified boundary.
Then the face poset is quasi-graded + Eulerian, where

$$\rho(x) = \dim(x) + 1.$$

$$\bar{\zeta}(x,y) = \chi(\operatorname{link}_y(x))$$

Open ?s and Work in progress:

1. Inequalities:

   Kalai convolution still works

   What about Ehrenborg's lifting technique?

2. $(P, \rho, \bar{\xi})$ Eulerian quasi-graded poset $\Longrightarrow$ Find W Whitney stratified space.

3. Combinatorial interpretation for cd-index coeffs.

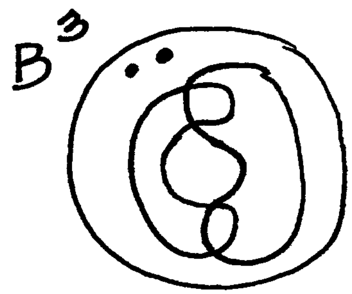
   [Purtill] n-simplex and n-cube

   [Karu] operators on sheaves of v.s.

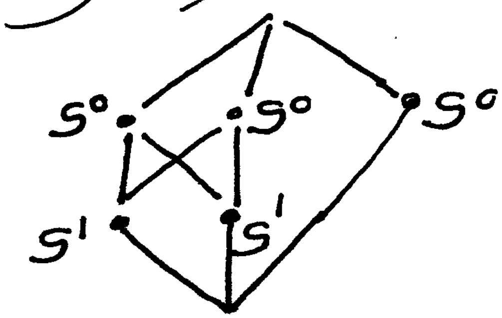
4. Stanley-Reisner ring for barycentric subdivision of a stratified space?
What should the Cohen-Macaulay property be?

5. Non-linear inequalities?

6. Inequalities for Ψ(manifold arrangements)?



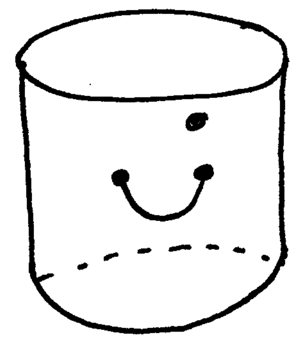
Intersection poset



7. Moci's gen'd Tutte polynomial for spherical + toric arrangements.
Develop a similar polynomial for manifold arrangements (or for some natural subclass).

Happy Birthday,

Bruce.

Φ () $= c^3 + dc - 2cd$

=

Φ (A birthday cake without candles ... since who is counting anyway ...)