PEEPS INTO INDIA'S MATHEMATICAL PAST

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JOHNSON has said that every man's life may be best written by himself. Even so, the history of any nation's achievements can best be written with insight and appreciation only by a discerning individual of that nationality. The history of Indian mathematics suffers from the defect that foreigners were the first to write it and promulgate their views about it to the world at large. Fortunately, the earliest students of Indian mathematics were first-rate mathematicians like Laplace, Chasles, Colebrooke and Hankel, who have written unstintingly their praise of Indian achievement. To quote Laplace, " It is India that gave us the ingenious method of expressing all numbers by means of ten symbols, each symbol receiving a value of position as well as an absolute value; a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity, the great ease which it has lent to all computations, puts our arithmetic in the first rank of useful inventions; and we shall appreciate the grandeur of this achievement the more, when we remember that it escaped the genius of Archimedes and Apollonius, two of the greatest men produced by antiquity." But the opinions of these earlier writers are now thrown to the background and overlaid by the recent onrush of lesser men who with their characteristic want of understanding, charity, forbearance and humility little realise that as historians they are dealing with the work of persons many of whom possessed an amplitude and genius far beyond their (historians') own.

Witness Dantzig[1] making the appalling error that *Lilavathi* is a treatise on general theology written in the eighth century of our era and remarking in the fullness of his wisdom that ' the Hindus may have inherited some of the bare facts of Greek science, but not the Greek critical acumen. Fools rush in where angels fear to tread..... They played with number and ratio, zero and infinity as with so many words.....' Witness again another prolific writer, Coolidge[2], who finds it hard to make up his mind as to the importance of Brahmagupta's work and says that it is hard to be patient with one who mixes truth and error so freely. These writers, who are themselves adept mixers of truth and error, write perforce secondhand making several mistakes of transcription and echoing the views of earlier propagandists like Al Biruni and Kaye who had nothing but contempt for Indian mathematics.

In books on History of Mathematics written in Europe and America the reference to Asiatic Mathematics is naturally meagre, though, historically, this mathematics has been the immediate inspiration of all medieval European mathematics and the progenitor of all modern undergraduate arithmetic, algebra and mensuration. In the latest edition of Archibald's *Outline of History of Mathematics*, the mathematics of the ancient Hindus is disposed of in less than a couple of pages with a bare mention of a few trivial results offered as it were as random samples.

How far even some of the later European researches have been anticipated by the ancient Hindus will never be understood, if not often misunderstood by non-Indians excepting a few gifted American historians like David Eugene Smith and George Abraham Miller. It is therefore deemed necessary to set forth in the following pages a brief account of India's mathematical past with special reference to contributions in advanced mathematics, which are anticipations of modern work. The mathematical concepts of periodicity, limits and proportion were fundamental in early Indian mathematics. The principle of proportional parts, tabulation of first and second differences and interpolation, linear and quadratic, with reference to trigonometrical and astronomical tables have been profusely made use of by the first mathematicians of India, Aryabhata and Brahmagupta. The concept of the sine-function, the solution of indeterminate equations of the first and second degrees, and the method of iteration in approximation are first-class mathematical discoveries which any nation may be justly proud of.

The ancient Greek mathematics is fast receding into oblivion; Euclid and Apollonius are being forgotten; Diophantus has been completely ignored with his alphabetic numerals; Archimedes survives meagrely in Elementary Physics. All this has happened because Greek mathematics has been difficult and clumsy with its repellent logic lacking freshness and vitality. In contrast, we mention the qualities of daring originality, sublime poetry, brilliant intuition and pithy exposition pervading the pages of ancient Indian mathematico-astronomical texts which are absolutely modern and practical and deserve fuller and more general study in our schools and colleges. This hidden lore submerged for historic reasons deserves to be salvaged and reinherited, as it were, by young India.

We confine ourselves to nine classics consisting of three texts and six authors, covering a period of two millennia from 800 B.C. to 1200 A.D.

I. Vedanga Jyautisha

The *Vedanga Jyautisha* is avowedly the oldest Indian work extant in Astronomy. According to the most modest estimate, its date is about

1200 B.C. The author is unknown, but the contents are derived from the sayings of a sage Lagadha who was well versed in affairs of the Calendar. The Hindu Calendar, ever since its inception, has been a complicated one involving the positions of the Sun, the Moon and the Nakshatras. The time of performing the sacrifices was regulated by this Calendar.

Even in the dim past, *Vedanga Jyautisha* recognised the importance of accurate calculation and gave the first place to Ganita:

[3] यथा शिखामयूराणां नागानां मणयो यथा ।
तद्वद्वेदाङ्ग शास्त्राणां गणितं मूर्धनिस्थितम् ॥ (*Verse* 4)

It requires a certain amount of advance in mathematics to be able to appreciate its importance in theory and practice. Indeed, the author of the *Jyautisha* shows a true understanding of the concepts of periodicity, concurrence and proportion. In less than two hundred short lines, he expounds a Calendar based on a cycle (Yuga) of five years beginning with the Winter Solstice and the first Tithi immediately after the New Moon of Magha, just like the modern civil year which begins with January 1, about a week after the Winter Solstice. The coincidence of solstice and tithi is supposed to occur in 1,830 days. This period is equated to 61 mean solar months, 66 lunar months, 67 nakshatra months and 124 parvas or semilunations. The discrepancies in the Calendar based on these data never amount to more than a day or two and were duly rectified by the priests by observation. Even to-day such adjustments in Jewish and Muhammadan Calendars are effected on the ocular testimony of persons who declare on oath that they have seen the first crescent of lunar phase on a particular evening, because they are not sure of their astronomical calculations for the New Moon.

The first European scholars who tackled this work were Weber and Thibaut and those portions which were obscure to them have now been correctly interpreted by the famous savant of Mysore, the late Dr. R. Shama Sastri in his recent edition,[3] with his own translation (1936). Dr. Sastry was particularly fortunate in discovering that the rules of the text were quite similar to those in ancient Jaina astronomical texts, like *Suryaprajnapti*, *Jyotishkaranda* and *Kalalokaprakasa*. There is some interesting mathematics in (1) the calculation of relative lengths of day and night in a nycthemeron, the lengths of the longest and shortest days being 18 and 12 muhurthas, equivalent to $14\frac{2}{5}$ hours and $9\frac{3}{5}$ hours respectively (corresponding to latitude of Kashmir) and (2) the positions of the moon in different parts of the year and the time of day when a parvan ends. Incidentally, the residues of multiples of 73, mod 124, are studied in some detail. This indicates early interest in number theory.

II. The Sulva-Sutras

The *Sulva-Sutras* are a set of mathematical rules for the construction of sacrificial altars prescribed in the Vedas. Every Vedic School had its own text of *Sulva-Sutras*. Of the extant texts, *Baudhayana Sulva-Sutras* are the oldest and the most comprehensive. The *Apastamba* rules are next in importance. The dates of these works are variously fixed between 800 and 500 B.C. Modern European scholars would bring them down to 200 A.D.

In modern terminology, the *Sulva-Sutras* are manuals of Practical Geometry based on the use of the Sulva or cord, which serves both as a ruler and as a pair of compasses, to transfer distances and to draw circles. The constructions relate to altars (technically, Viharas), of various forms and of given dimensions. Thus a Mahavedi (अष्टविंशत्यूनं पदसहस्रं महावेदिः)[4] is an isosceles trapezium of area 972 sq. units with 24 and 30 units of length for parallel sides and 36 units of length for altitude, while an Aswamedhika Vedi is a similar Vedi of twice the area.

The Vedic sacrificial priests have to construct squares, circles, trapezia, and combinations of these, of given area and enlarge or reduce them in given proportions. The *Sulva-Sutras* had, therefore, to tackle at the outset three fundamental problems: (1) the construction of the right angle, (2) squaring the circle and its inverse, and (3) evaluating a quadratic surd like $\sqrt{2}$. The first problem was solved by discovering such triplets as (3, 4, 5); (5, 12, 13); (8, 15, 17); (7, 24, 25); (12, 35, 37) and their multiples, which are the sides of right-angled triangles and were specially adapted to rope-manipulations. The Pythagorean theorem was envisaged as a property of the rectangle, the square on either diagonal being equal to the sum of the squares on two adjacent sides. The two other problems were tackled by arithmetico-geometrical methods. Apasthamba directs the squaring of the circle and the circling of the square in the following terms:

[4] चतुरश्रं मण्डलं चिकीर्षन् मध्यात्कोट्यां निपातयेत् । पार्श्वतः परिकृष्यातिशयतृतीयेन सहमण्डलं परिलिखेत् सानित्यामण्डलं । यावद्धीयते तावदागन्तु ॥

मण्डलं चतुरश्रं चिकीर्षन्विष्कम्भं पञ्चदशभागान् कृत्वा द्वावुद्धरेत् । त्रयोदशावशिष्यन्ते सा नित्या चतुरश्रम् ॥

This is equivalent to $d = a + \frac{1}{3}a(\sqrt{2} - 1)$, inversely $a = d(1 - \frac{2}{15}) = \frac{13}{15}d$, a being the side of the square and d the diameter of the equivalent circle. These are based on $\pi = 3$ and $\sqrt{2} = \frac{3}{2}$ as first approximations. The quadratic surd $\sqrt{2}$ is disposed of thus:

समस्य द्विकरणी । प्रमाणं तृतीयेन वर्धयेत् तच्चतुर्थेनात्मचतुस्त्रिंशोनेन सविशेषः ॥[4]

obtained by the usual method of root extraction

$$\left(\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} \text{ approx.}\right).$$

Vedic constructions with specific types of bricks led to quadratic equations and linear simultaneous indeterminate equations in as many as five variables. On the whole, it may be said that the *Sulva-Sutra* authors had faced more difficult problems at the dawn of mathematics than many of us to-day.

Unlike the Greek Gods, the Hindu Gods were more reasonable and sympathetic towards the mathematical difficulties of their devotees and accepted their humble approximations, taking the will for the deed.

III. Surya Siddhanta

The *Surya Siddhanta* is not only the most popular but also the most authoritative Indian work on Astronomy. While other Siddhantas are human revisions of the earlier semi-divine texts, this alone claims pure celestial inspiration untouched by human interference. The text, however, has passed through several recensions from 500 A.D. to 1000 A.D., but quite unobtrusively. It looks as if the text had revised itself through the ages. As the text never fell into disuse like, for example, the *Brahma Siddhanta*, no officious astronomer* was needed to rescue it from oblivion. Therefore it still possesses the pristine purity and sanctity of its original author Maya, whom the European scholars wish to identify with one of the Ptolemies of Alexandria. It is free from superfluous mathematics and written in austere scientific style with no extravagant poetry. It compares favourably with any modern elementary text-book on astronomy, if we omit the innovations of the last three centuries. The text comprises fourteen chapters, of which only one, *i.e.*, the eleventh is astrological in character. The topics dealt with are, in order, mean motions of the planets, the Sun and the Moon, true places, co-ordinates and time, eclipses, parallax, conjunctions, asterisms, heliacal risings and settings, Moon's rise and set, cosmogony, astronomical instruments, and the different modes of reckoning time. Though it is a far cry from an astronomical text-book used in our colleges to the training required for the computations of the Nautical Almanac, a thorough knowledge of the *Surya Siddhanta* will enable one to calculate the Indian Ephemeris straightaway.

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* Kamalakara of the 17th century professes to rescue the doctrines of *Surya Siddhanta* from bad commentators. He is led away by mere jealousy. His emendation is uncalled for and unnecessary.

Astronomical instruments are mysteries. Their secret should be carefully guarded lest the astronomers' job should be, otherwise, insecure. In ancient times, obscurity of exposition was the only safeguard against plagiarism. The advice given in the *Surya Siddhanta* to the unwary astronomer was:

गोप्यमेतत्प्रकाशोक्तं सर्वगम्यं भवेदिह ।

So, merely the names of instruments are mentioned with half-hearted explanations.

The boast of 'the illustrious and unfortunate' Frenchman Jean Sylvain Bailly who published the 'Traite de l'astronomie Indienne et orientale' (1787) was that no European would ever decipher the *Surya Siddhanta*. The effect of it was that Davis made an analysis of the *Surya Siddhanta* in 1789 and Rev. E. Burgess[5] published his famous translation in 1860, of which the Calcutta University has now published a reprint. We give here a specimen of *Surya Siddhanta* style and Burgess's translation:—

[6] त्रिंशत्कृत्योयुगेभानां चक्रं प्राक्परिलम्बते ।
तद्गुणाद्भूदिनैर्भक्ता द्युगणाद्यदवाप्यते ॥
तद्दोस्त्रिघ्नाद्दशाप्तांशा विज्ञेया अयनाभिदाः ॥

"In a Yuga, the circle of asterisms falls back eastward thirty score revolutions. Of the result obtained after multiplying the sum of days by this number and dividing by the number of natural days in an age (Yuga), take the part which determines the sine, multiply it by 3 and divide by 10; these are the degrees of Ayana."[5] Decoded, it means that the equinoxes shift 54″ per year.

IV. Aryabhata

Aryabhata is the first worldly figure (लौकिक) associated with ancient Hindu Mathematics after the eras of supermen, Rishis, and sages of the hoary past. He was both a founder and a reformer, the first scientific man to give a form and an individuality to the scattered bits of mathematics that existed before his time, the first to classify and set forth three things—(गणित) mathematics, (कालक्रिया) the reckoning of time, and (गोल) spherical astronomy. His work, designated as the *Aryabhatiyam*, was produced in 499 A.D., when the author was only twenty-three years old. According to Neelakanta a famous Kerala commentator, Aryabhata belonged to the country called Asmaka, identified by some as the ancient Travancore. But Aryabhata must have migrated to Kusumapura, the flourishing capital of the Guptas of the fifth century A.D. and probably he produced his work under their patronage. He must have been a self-made man, for he refers to no Guru and claims to have discovered true knowledge by his own efforts (स्वमतिनावा). He was an advocate of the *Brahma-siddhanta.* The first printed edition[7] of the *Aryabhatiyam* with the commentary *Bhatadipika* of Parameswara was due to Dr. Kern of Holland (1874) and ever since, there have been several notes and translations by various authors, Indian, European and American. The text comprises 121 (apparently only 108) short couplets. Of these 10 deal with Tables codified in a jargon of which the key is that क to म represent 1–25, य to ह, 3–10 each multiplied by 10, and अ to औ denote powers of 100 from 1 to 100^8, the long and the short vowels of the same kind not being distinguished. Thus, ङि.शिवुण्ऌख्षृ = 15,82,23,7,500; the consonants behave like detached coefficients of a polynomial involving powers of 100. A namesake of Aryabhata, coming half a millennium later, introduced another scheme of notation, in which the consonants are given values according to the scheme क to ञ, ट to न each denoting 1–9 and 0, प to म, 1–5, य to ह, 1–8 and letters are read from left to right. This scheme is called Katapayadi, which is more popular with letters read from right to left. For example, in *Sadratnamala* (1830), $\pi \times 10^{17}$ = भद्राम्बुधिसिद्धजन्मगणितश्राद्धाश्मयद्भूपगीः

= 314159265358979324 (read in the reverse order).

The important results due to Aryabhata are:—

(1) The concept of the sine of an arc as the semi-chord of twice the arc;

(2) the recurrence formula $\Delta^2 \sin n\alpha = -\dfrac{\sin (n+1)\alpha}{3438 \sin \alpha}$ approximately, in modern notation when $\alpha = 225'$ and n (a positive integer) < 24;

(3) the recognition of the method of inverse operations, which reminds one of Jacobi's famous direction 'invert' the keystone of all mathematical research;

(4) the recognition of the rotation of the earth on its axis;

(5) the correct theory of the eclipses;

(6) a reference to secular perturbations in the 9th verse of the *Kalakriya* section:

[7] उत्सर्पिणीयुगार्धं पश्चादवसर्पिणी युगार्धं च

मध्ये युगस्य सुषमादावन्ते दुष्षमेन्दूच्चात् ॥

which may be compared with a modern statement 'The sidereal month is slowly becoming shorter and will continue to do so as long as the eccentricity of the earth's orbit continues to diminish. After about 24,000 years, the effect will diminish to zero and then become reversed';

(7) Kuttaka which is literally (repeated breaking) and technically the same as 'continued fraction', applied to the problem of finding a number which leaves given residues when divided by certain numbers. A numerical ancestor of this problem can be traced to a Chinese mathematician of the first century A.D.; but the complete and general solution was the one given in the last two couplets of the *Ganitapada*, where a rule is given for successive reductions to simpler cases until the solution can be guessed. In spirit, this method is analogous to the famous Method of Descent, due to Fermat.

V. Varahamihira

Varahamihira is the Prince of Indian Astronomers and enjoys, even to-day, the highest esteem and respect, not only of the lay Hindus, but of foreign Oriental scholars, who are pleased to find in his works explicit laudatory references to Greek scholarship. He possessed the rare gift of transcending all jealousies and could appreciate whatever was good in the different astronomical systems current in his time. His was an encyclopædic mind which comprehended all science, all art and all literature of his time. He expounded in his *Brihatsamhita* all topics under the Sun from perfumery to meteorology and seismology. He wrote four other works *Brihatjataka, Brihatyatra, Brihatvivahapatala,* and *Panchasiddhantika*, of which the last is of great historical and mathematical interest.

He imbibed his learning from his father Adityadasa in the village of Kapittaka and later settled in Avanti. He was a Vaishnavite and a worshipper of Sri Rama. In the opening chapter of *Panchasiddhantika*, there is a reference to the date, first tithi, sukla, in the month of Caitra of 427 Saka (505 A.D.). From sunset on this date is to be reckoned the number of civil days elapsed, according to Paulisa and Romaka Siddhantas. Some authorities consider this as the date of birth of Varahamihira or the date of composition of his work. But the date fixed for astronomical calculations must possess some astronomical advantages. Very likely, it is the date of zero ayana. In modern times, a similar date, *viz.*, the equinox of 1950 A.D. approved by the International Astronomical Union of 1928 is fixed for the calculation of the Ephemerides of minor planets. Anyhow, there is no doubt that Varahamihira lived in the sixth century and preceded Brahmagupta.

He was an advanced astronomer who could compute the eclipses and the positions of the planets. He could determine the moment when the Sun crosses the prime vertical. Indeed he says:

[8] सममण्डल लेखा संप्रवेशवेलां करोतियोर्कस्य ।
तत्प्रत्ययं च जनयति जानाति सभास्करं सम्यक् ॥
वर्षेणभगणमर्कौ यदिभुङ्क्ते किं ततो यथेष्टदिनैः ।
अज्ञोप्यवं गणयति किं नरविं लोष्टरेखाभिः ॥

'That person who determines the moment at which the Sun crosses the prime vertical and who is able to produce general confidence in his calculations, thoroughly understands the Sun, whereas even an ignorant fellow can easily answer, mechanically by means of marks made with a piece of chalk, such questions as to find the number of degrees traversed by the Sun in a given number of days, according to mean motion.'

Varahamihira's π was, however, the traditional $\sqrt{10}$. He calculated his Table of Sines by the neat trigonometrical formula
$\sin^2 r = (\frac{1}{2}\sin 2r)^2 + (\frac{1}{2}\text{vers } 2r)^2$, which he applied to 60°, 45° and 30° and their halves and so on, and got down to 3° 45′, its multiples, and their complements. Two of his works were translated into Arabic by Alberuni (*c.* 1000 A.D.). According to some Oriental scholars, he died in 587 A.D.

VI. Brahmagupta

Brahmagupta was the greatest mathematician of India before 1000 A.D. He well deserves the appellation 'Ganakachakrachudamani' given to him by Bhaskara. Unlike Aryabhata, he has a clear exposition though he indulges occasionally in cryptic statements (Chapter 20, *Brahmasphuta Siddhanta*). The Arabs learnt astronomy from Brahmagupta's text before they became acquainted with Ptolemy. He is supposed to have lived and worked in Ujjain, where Varahamihira also worked.

Born in 598 A.D., Brahmagupta, the son of Jishnu, wrote the famous *Brahmasphuta Siddhanta*[9] in his thirtieth year, when Vyaghramukha was the ruling Saka king. This work comprises twenty-four chapters of 1008 Arya couplets. It may be remarked that the Hindus are fond of the numbers 108, and 1008, and that Aryabhata, the greatest rival of Brahmagupta, wrote his work in 108 couplets. The mathematical chapters are numbered 12, 18 and 19, while the sine-tables are discussed in a section of the 21st chapter on Spherics. Numbers are represented according to the popular system of word numeration in which a word represents the number of component units of the collective entity signified by the word; thus (दन्त) 'teeth' represents 32. It may be worth while to compare the underlying principle of this notation with Bertrand Russell's definition: "A number is any collection which is the number of one of its members."

Brahmagupta devotes a special chapter, *viz.*, 11th, to a critical review of other astronomical works. The special target of his attack was Aryabhata, whom he condemned outright as knowing nothing:

जानान्येकमपि यतो नार्यभटः गणितकालगोलानाम् ।

The attack is unjustifiable and prompted by bigotry. It is a pity that the history of mathematics should be tainted by such unreasonable animosities, witness those of Steiner towards Plücker, Hooke towards Newton, Kamalakara towards Bhaskara and so on.

Two other works attributed to Brahmagupta are *Dhyanagrahopadesa* and *Khandakhadyaka,* manuals on astronomical computation. It is strange that in the latter work, written towards the close of his life, the author should have mellowed and deemed it an honour to write a work comparable in merit to Aryabhata's.

We mention four outstanding mathematical results of Brahmagupta in modern notation:

(1) The Principle of the Composition of Quadratic Forms, which states that:

If $x = a$, $y = b$ satisfy $\alpha x^2 + p = y^2$
and $x = c$, $y = d$ satisfy $\alpha x^2 + q = y^2$,

then $x = bc \pm ad$, $y = bd \pm \alpha ac$ satisfy $\alpha x^2 + pq = y^2$;

a result rediscovered by Euler *c.* 1000 years later, after spending much labour and thought. In particular, if $x = p$, $y = q$ satisfy $Ny^2 - 4 = x^2$, N a positive nonsquare integer, then

$$x = (p^2 + 2)\left\{\frac{(p^2 + 3)(p^2 + 1)}{2} - 1\right\},\ y = pq\,(p^2 + 1)\ (p^2 + 3)/2$$

satisfy $Ny^2 + 1 = x^2$.

(2) The quadratic interpolation formula, given apparently as applicable to the sine-function:

$$\sin(x - h\theta) = \sin\ x - \frac{\sin\ (x + h) - \sin(x - h)}{2}\,\theta + \frac{\sin\ (x + h) - 2\sin\ x + \sin\ (x - h)}{2}\,\theta^2,\ (0 \leqslant \theta \leqslant 1)$$

or $u_x = u_0 + \frac{1}{2}x\,(\Delta u_0 + \Delta u_{-1}) + \frac{x^2}{2!}\Delta^2 u_{-1}$, a precursor of Stirling's formula.

(3) The formulæ in Trigonometry:

$a = 2R \sin A$; $bc = 2R \times$ alt. from A;

area of a cyclic quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$

and the squares of the diagonals in the form pq/r, qr/p

where $p = bc + ad$, $q = ca + bd$, and $r = ab + cd$.

(4) The general solution of the indeterminate equation $ax - by = 1$ by means of the Kutta process, in essence equivalent to the modern Continued Fraction. According to Brahmagupta, a knowledge of Kutta is indispensable for Acaryapadavi (corresponding to the modern Doctorate).

VII. Sridhara

In Cajori's *History of Mathematics*, it is stated that Sridhara lived later than Mahavira and wrote the *Ganitasara*. In 1938, Avadesh Narayan Singh pointed out that the *Patiganita* of Sridhara discovered by him appeared to belong to a time prior to Brahmagupta (628 A.D.). However, the *Trisatika* of Sridhara as edited by Sudhakara Dwivedi in 1899 was the basis for a translation into English by N. Ramanujacharya,[10] of Pachayappa's College, Madras, in 1912, with Notes by G. R. Kaye. This text contains 65 couplets which are accompanied by answers to questions elucidated with symbolic statements in the manner of Euclid's particular enunciations. In the evolution of expository mathematics of the ancient Hindus, we find the following four well marked stages progressing steadily towards clarity and popularity:

(i) Mere rules without any explanations or illustrative problems, as in Aryabhata; no problems are given at this stage;

(ii) Rules and problems as in Brahmagupta;

(iii) Rules and problems with expository symbolic statements and answers;

(iv) Rules and problems with complete solutions as in Bhaskara.

If this order of evolution be accepted, *Trisatika* ought to come later than Brahmagupta. Probably it belongs to the eighth century.

The content of *Trisatika* is mainly arithmetic with series, mensuration and shadow-problems. In mensuration, irregular figures are dealt with as made up of quadrilaterals and segments of circles. The ratio of a circumference of a circle to its diameter is the popular Indian value $\sqrt{10}$. It is stated that good approximations to square roots can be obtained by multiplying the given non-square quantity by a large square, taking the integral part of the square-root and dividing by the square-root of the factor selected.

This rule gives away the secret of the Indian approximations to π, *viz.*, 3, $3\frac{1}{6}$, $3\frac{1}{7}$, $3\frac{3}{20}$ obtained by means of the multipliers 1, 36, 49, 400 respectively and taking the integral parts of the square roots of 10, 360, 490, 4000. Here we have an explanation of the Indian rediscovery, though by a semi-legitimate process, of the much advertised Archimedean approximation $3\frac{1}{7}$.

Sridhara does not stint from numerical and empirical approximations, just in the manner of Sulvasutrakaras. He gives $\sqrt{\frac{10}{9}\left\{\frac{a\,(c+a)}{2}\right\}^2}$ as the area of a segment of a circle, a being the arrow and c the chord.[10] This value is not given either by Brahmagupta or Bhaskara. Elsewhere, we find in *Trisatika* the remark that the integral part of the square root of $n^2 + n$ is n. This is rather unusual in Indian mathematics, nay even in our under-graduate mathematics; another similar example occurs in Devaraja's *Kuttakara Çiromani*, a specialised work on Kuttaka, of a much later date: 'The integral part of the cube root of three times the sum of the first n square integers is n.' These examples are perhaps the ancestors of Ramanujan's identities:

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+2}],\ [\tfrac{1}{2} + \sqrt{n+\tfrac{1}{2}}] = [\tfrac{1}{2} + \sqrt{n+\tfrac{1}{4}}].$$

(vide *Collected Papers of Ramanujan*, p. 332.)

VIII. Mahavira

Mahavira is one of those important men of scientific attainments who bridge the gulf of five centuries between Brahmagupta and Bhaskara. His work *Ganitasarasangraha* is an index of the development of mathematics in the expository and symbolic phases, as it contains classification of operations, statement of rules and varieties of problems. The only addition in content is 'Permutations and Combinations'. The operation of Kuttaka or continued pounding which is the same in spirit and sense as the modern continued fraction is known thoroughly in this period and extended to simultaneous indeterminate equations of the first degree in any number of variables. One may roughly put the date of Mahavira midway between Brahmagupta and Bhaskara, possibly a few decades nearer to the former than the latter. Mahavira professed the Jaina religion and probably lived at the court of one of the old Rashtrakuta monarchs Amoghavarsha Nripatunga, who ruled the region now called Mysore. The Jains are particularly fond of mathematics especially puzzles which baffle one at first sight. It is no wonder that the *Ganitasarasangraha* teems with them. Here is a sample. A courtezan has five suitors, of whom she likes only three. But she says to each one of them that she likes him alone. How many are truths?

In profundity and profusion of problems and their poetic setting Mahavira has no rival. Even Bhaskara must be a second to him. It is sometimes difficult to say whether you enjoy the solution more than the delicious setting. In Chapter VI, Mixed Problems, thirteen problems on indeterminate equations are set with the background of a luxurious forest scene full of fruits and flowers of all sorts—jambus, lime trees, plantains, areca palms, jack trees, palmyras, punnaga trees and mango trees. Elsewhere a person offers flowers to all Jaina deities according to an arithmetical scheme and returns home empty-handed but full of piety and sanctity. Indeterminate problems in Geometry are of the following type: 'There are two isosceles triangles. The area of the first is twice that of the second and the perimeter of the second is twice that of the first. What are the values of the sides of the two triangles?'

In mensuration, Mahavira indulges in bold empirical approximations for the areas and volumes of all sorts of irregular and curvilinear figures. In particular, his formulæ for the perimeter and the area of an ellipse are interesting.

[11] व्यासकृतिष्षड्गुणिता द्विसङ्गुणायामकृतियुता (पदं) परिधिः ।
व्यासचतुर्भागगुणश्चायतवृत्तस्य फलम् ॥

If a, b are the semi-axes of the ellipse, $(a > b)$, then the perimeter is $\sqrt{24b^2 + 16a^2}$ and area $\sqrt{24b^2 + 16a^2} \times 2b/4$.

These formulæ reduce to the correct formulæ for the circle if we take $\pi = \sqrt{10}$ as Mahavira does. Assuming then π for $\sqrt{10}$, the formula for the perimeter may be written in terms of π, a, e (eccentricity), as $2\pi a \sqrt{1 - \frac{3}{8} e^2}$ which is very much better than the formula in Clark's *Mathematical Tables* (p. 24) and compares not very unfavourably with the modern value $2\pi a (1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 + \ldots)$. But the formula for the area is hopelessly wrong. If we remember how Ramanujan also amused himself with empirical values for π and the perimeter of an ellipse, we are tempted to remark on the unique Indian taste for approximations which having started with the construction of the altars seems to persist through the ages.

It will be no exaggeration, in fine, to say that Mahavira illustrates in his own personality the eight characteristics of a good mathematician delineated by himself: *Quickness, Insight, Reasoning Power, Brilliance, Correct Comprehension, Memory, Originality, and Algebraic Intuition.*

IX. Bhaskara

The most renowned Indian mathematician and astronomer before the advent of modern European Science was Bhaskara. He was born in 1114

A.D. in the village of Bijjadabidu, near the foot of the Sahya Hills in Deccan. He derived his education from his father Maheswara who was well versed in all ancient lore and greatly honoured and respected in his time. Like other astronomers, he also worked at Ujjain. His fame travelled down the ages and roused many jealousies, the most notable being that of Kamalakara of Benares, a great admirer of Ulugh Begh, the famous Persian astronomer-prince. Though Bhaskara also criticized his predecessors, he did it in the interests of his subject and with proper humility. Otherwise, as he says, he would not get proper credit for his own utterances. In his own words,[12]

[12] कर्तव्ये स्फुटवासनाप्रकथने पूर्वोक्तिविश्वासिनां तत्तद्दूषणमन्तरेण नितारां नास्तिप्रतीतिर्यतः ॥

In his thirty-sixth year (1150) he wrote the famous *Siddhanta Çiromani*, the crown and cream of siddhantas or the last word in astronomical science, as it then stood. It comprises four parts, Arithmetic under the title *Lilavati* suggestive of the sweet maidenly freshness of the work, *Bijaganita* or Algebra (may not 'geb' in algebra be merely the Arabic inversion of the Sanskrit बीज ?), Ganita or astronomical calculations, in other words, Practical Astronomy, and Gola or spherical astronomy. In this great work marked as much for its poetic eloquence as for its mathematical profundity, the ideas started by Aryabhata and developed half-way by Brahmagupta reached their completion and perfection. The most notable of these ideas relate to indeterminate equations and the methods developed for solving them in Kuttaka, Varga-prakriti, and Cakravala† which last leads to a type of continued fraction unknown to Europe till 1873. In astronomy he worked out the idea of instantaneous motion and incidentally discovered that the differential of $\sin\theta$ is $\cos\theta\, d\theta$. *Lilavati* is very much like a modern text-book on Elementary Mathematics. It includes algebra, geometry, mensuration, heights and distances, progressions, permutations and combinations and indeterminate equations. Directed numbers, quadratic surds, equations, simple and quadratic, indeterminate equations in several unknowns represented by colours, and puzzle problems form the subject-matter of *Bijaganita* or *Avyakta Ganita* (calculation with unknown elements). It is worthy of note that negative numbers and irrationals are classified among Avyaktas (those which are imperceptible to the senses). Division by zero is elucidated with a scientific and religious zeal characteristic of a devout Hindu Scientist, and 'infinity' is assimilated to the nature of God unperturbed by finite additions and subtractions. Any number a divided by 0 is just $a/0$, Khahara (खहार)

† A detailed study of a sequel to Bhaskara's Cakravala or cyclic method is published in an early issue of this Journal, under the title "Theory of the Nearest Square Continued Fraction" (Vol. I, Parts II and X, Section A).

characterised by two properties, (i) that it is no number and (ii) that it is unaffected by finite operations.

[13] "अस्मिन्विकारः खहरेनरशावपि प्रविष्टेष्वपि निःसृतेषु ।
बहुष्वपिस्याल्लयसृष्टिकालेऽनन्तेऽच्युतेभूतगणेषुयद्वत्" ॥

Bhaskara's solution of a stray biquadratic equation contains a valuable hint for a general solution, leading to the same auxiliary cubic as Descartes' (1637). It is interesting to note that a challenge problem of Fermat (1657) is fully worked out by Bhaskara to illustrate that it is a favourable case where the solution is immediately in sight, if we apply the principle of composition of forms. Thus, there are so many anticipations of the mathematics of the seventeenth century that a Kaye of 3,500 A.D. may well fix Bhaskara at 1880 A.D. conveniently after Lagrange.

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