Notes and Questions.

New Light on Bhaskara's Chakravala or Cyclic Method of solving Indeterminate Equations of the Second Degree in two Variables.

1. Introduction.

The equation $x^2 - Ny^2 = 1$* has a long and interesting history† behind it of more than 2000 years. The earliest civilized nations of the world, the Indians and the Greeks, were fascinated by the problem. While the latter never gave more than particular solutions for some simple cases like $2x^2 - y^2 = \pm 1$, the former were the first to realize the true inwardness of the problem and to give a general solution based on the principle of composition of quadratic forms. More than thousand years before Euler re-discovered, after spending much thought and labour on the problem, what he calls *the following remarkable theorem* which contains within it the foundation of higher solutions:

"If $x = a,\ y = b$ satisfies $\alpha x^2 + p = y^2$,

and $x = c,\ y = d$ satisfies $\alpha x^2 + q = y^2$,

then $x = bc \pm ad,\ y = bd \pm \alpha ac$ satisfies $\alpha x^2 + pq = y^2$"

we find Brahmagupta dealing with the same theorem in a special chapter 'Vargaprakriti' and drawing important corollaries from it. In particular, he derives the identity

$$\alpha \left(\frac{2\,ab}{p}\right)^2 + 1 = \left(\frac{b^2 + \alpha a^2}{p}\right)^2 \text{ from } \alpha a^2 + p = b^2.$$

Special identities are also given by him for deriving integral solutions of $Ny^2 + 1 = x^2$ from those of $Ny^2 \pm 4 = x^2$.

In the period between Brahmagupta (born 598 A.D.) and Bhaskara

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* It is understood throughout this paper that N is a non-square integer.

† There is a whole book '*Geschichte der Gleichung* $t^2 - Du^2 = 1$' devoted to this history by H. Konen. It contains a comparison between Lagrange's procedure and the Indian. The book, however, has not been accessible to me for direct reference.

(born 1114 A.D.), there must have taken place some modifications* on Brahmagupta's work, of which, however, we have little knowledge, except from the indirect references thrown out by Bhaskara. About 1150 A.D., Bhaskara surprises us with his brilliant method for integral solutions, designating it as 'Chakravala' after the ancients. Thus, indirectly Bhaskara throws the hint that he does not claim it as his own. But it is certainly not Brahmagupta's. Owing probably to the fact that the mathematical writings of Brahmagupta and Bhaskara, separated though they are by about five centuries, are put together in the same volume by H. T. Colebrooke, the credit that is really Bhaskara's is sometimes shared by Brahmagupta also. Referring to the cyclic method, T. L. Heath says (on p. 281 of his *Diophantus*): "If the Greeks did not accomplish the general solution of our equation, it is all the more extraordinary we should have such a general solution in practical use among the Indians as early as the time of Brahmagupta under the name of the 'Cyclic method'; ... ...

... ...

"The 'cyclic method' of solving the equation $x^2 - Ay^2 = 1$ is found in Brahmagupta and Bhaskara and is well described by Hankel, Cantor, and Konen."

Thus, the cyclic method is wrongly attributed to Brahmagupta also. In the absence of explicit reference to this method in the works of the earlier writers, it is fair to take it as Bhaskara's; for there is an element of doubt in the interpretation of 'चक्रवालमिदं जगुः' and Bhaskara may be after all giving credit to his predecessors merely for the name 'चक्रवाल' as applied to all iterative operations in mathematics.

The true nature of the Chakravala has not been understood by many eminent authorities from the time of Colebrooke onwards. This accounts for its neglect even by Colebrooke who speaks so highly of one of Bhaskara's rational solutions which happens to be exactly the same which Lord Brouncker devised to answer a question proposed by way of

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*As an evidence of such modifications may be noted the fact that the technical terms 'आदि' and 'अन्त्य' used by Brahmagupta for the two variables (y, x) in the equation $Ny^2 + 1 = x^2$ have changed by Bhaskara's time to 'ह्रस्व' and 'ज्येष्ठ.'

challenge by Fermat in 1657. H. J. Smith in his reports* on the Theory of Numbers, remarks, probably on the authority of Colebooke, that Lord Brouncker and Wallis first gave rational solutions just like Bhaskara and Brahmagupta 700 years before them, having misunderstood the nature of the problem. Surely Bhaskara and Brahmagupta did not misunderstand the problem, like Lord Brouncker and Wallis. The Indians never proceeded along the superficial lines adopted by Diophantus in his Lemma to Book VI, 15 (*vide* Heath's *Diophantus*, p. 238) but showed their grip over the problem by enunciating the important formula, which is equivalent to tho composition of quadratic forms rediscovered by Euler, a thousand years later. Again, T. L. Heath himself, on p. 285 of his *Diophantus*, holds that the Indian method is the same as that rediscovered and expounded by Lagrange in his memoir of 1768. He is probably inclined to this view through H. Konen's comparison of the two methods—the Indian and the Lagrangean—in his book on the history of the equation.

One may also mention in this connection the definitely contemptuous view of the late G. R. Kaye in several of his articles. In the article on 'The Source of Hindu Mathematics' (J. R. A. S. 1910), he writes:

"Indeterminate equations play an important part in Hindu mathematics and the discovery of solutions of $Du^2 + 1 = t^2$ in Hindu works of a fairly early date was considered very remarkable. Possibly the fact that Fermat, Wallis, Brouncker, Euler, Lagrange, and others paid considerable attention to the problem, gave the discovery of it in Hindu works *a somewhat fictitious value*. ... ...

Bhaskara gives *some* alternative methods for the solution of the Pellian Equation but *in no essential does he improve on Brahmagupta*."†

Again, on p. 672, *East and West*, (Vol. XVII, No. 201, July 1918) we have such statements as:

"The discovery of Hindu works on rational, integral solutions of the so-called Pellian Equation $ax^2 + 1 = y^2$ made something of *a splash*.

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* *Vide Collected Mathematical Papers* of H. J Smith (Edited by J W. L. Glaisher) Vol. I pp 192—200.

† It will be easily conceded after reading this paper how Bhaskara's Cyclic Method is really an advance on Brahmagupta's.

The very problem had not long before been the subject of correspondence and discussion in Europe and this gave emphasis to the discovery of the Hindu solution and the combination of events perhaps led to an over-estimate of the value of the Hindu contribution. More recent research leads to the conclusion that the very occurrence of these Hindu solutions is a result of Greek influence."

Lastly, it is unfortunate that even K. J. Sanjana (in the J. I. M. S. Notes and Questions, Vol. XVIII, No. 1, p. 22), refers to Bhaskara's method of solving the equation $x^2 - 67y^2 = 1$ as a process of converting $\sqrt{67}$ into a continued fraction and gives a table* of his own to show the reader that Bhaskara's process tallies exactly (this, of course, is not a fact) with that outlined by Chrystal.

It is the object of this paper to bring out at some length the true inwardness of Bhaskara's cyclic method and to point out that the Chakravala is not the same as either the continued fraction method or the crude method of Brouncker and Wallis. On the other hand, it will be shown in the sequel that Bhaskara's method is more in line with Gauss's solution based on his theory of Quadratic Forms. The present investigation will reveal that Bhaskara's method leads naturally to a new set of reduced forms distinct from those of Gauss, Lagrange, Klein, and Hermite. Further, Bhaskara's method does not lead straight to a simple continued fraction but to an irregular one. It has also the advantage of reducing the number of recurring elements and thus may claim a certain superiority to the other methods in the matter of practical computation.

2. The Chakravala or Cyclic Method explained.

Brahmagupta's formula for the composition of quadratic forms, *viz.*:

$$(XX' \pm \alpha YY')^2 - \alpha (XY' \pm X'Y)^2 = (X^2 - \alpha Y^2)(X'^2 - \alpha Y'^2)$$

is also repeated by Bhaskara. Both of them realise that with the help of this, the solutions of the equation

$$Ny^2 + 1 = x^2$$

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* Mr. Sanjana wrongly believes that his table represents Bhaskara's method. Bhaskara's solution agrees with Sanjana's table only as far as the first three steps and from the third step Bhaskara proceeds immediately to Sanjana's fifth step and from thence takes a short cut to the required solution.

can be derived easily from those of

$$Ny^2 \pm 2 = x^2, \text{ or } Ny^2 \pm 4 = x^2.$$

Bhaskara has attempted to go one step further to show that the roots of the equations

$$Ny^2 + 1 = x^2, \; Ny^2 \pm 2 = x^2, \; Ny^2 \pm 4 = x^2$$

can be derived, by successive reduction, from those of the more general equation

$$Ny^2 \pm k = x^2 \, *$$

where k is any integer. The method of such reduction is outlined as the *Chakravala*:

ह्रस्वज्येष्ठपदक्षेपान्भाज्यप्रक्षेपभाजकान् ॥
कृत्वा कल्प्यो गुणस्तत्र तथा प्रकृतितश्च्युते ।
गुणवर्गे प्रकृत्योनेऽथवाल्पं शेषकं यथा ॥
तत्तु क्षेपहृतं क्षेपः व्यस्तः प्रकृतितश्च्युते ।
गुणलब्धिः पदं ह्रस्वं ततोज्येष्ठमतोऽसकृत् ॥
त्यक्त्वा पूर्वपदक्षेपांश्चक्रवालमिदं जगुः ।
चतुर्द्व्येकयुतावेवमभिन्ने भवतः पदे ॥
चतुर्द्विक्षेपमूलाभ्यां रूपक्षेपार्थभावना ॥ †

Considering the smaller root, the greater root, and the additive (of the given equation) as respectively the dividend, addend, and divisor (of a linear indeterminate equation), the multiplier should be so determined that the square of this multiplier being subtracted from the given co-efficient or the co-efficient being subtracted from the square of the multiplier (as the case may be), the residue is the least (under the circumstances); this residue divided by the original additive is the (next)

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* In the Indian rule, N is called the co-efficient, y the lesser root, and x the greater root while ($\pm k$) is known as the addend or additive. The terms 'lesser' and 'greater' are not to be taken in their literal sense, just as the word 'imaginary' in modern mathematics should not be understood too literally.

† *Vide* p. 185, verses 46 - 50; *Bijaganita* of Bhaskaracharya with the Commentary of Durga Prasad Trivedi, 2nd Edn., Lucknow, 1917, hereafter referred to as Bijaganita.

additive, being reversed in sign if the residue be obtained by subtraction from the co-efficient. The quotient corresponding to the multiplier in the linear equation is the smaller root, whence the greater root can be obtained. This process may be repeated with the new additive, smaller and greater roots, in place of the old ones. Such an iterative process is styled *Chakravala*. By this method, we get integral roots for equations with additives ± 4, ± 2, ± 1, whence may be derived by the principle of composition, the roots of the equation with *unity* as the additive.

For a clearer understanding of the above rule, one has only to restate it in modern terms thus:

Let the roots of the equation

$$Ny^2 + k = x^2 \qquad \dots \quad (1)$$

be $x = a,\ y = b$ (N being a positive, non-square integer).

Solve the linear indeterminate equation

$$\frac{bx + a}{k} = y$$

in integers and choose a value l for x so that $| l^2 - N |$ may be the least possible.

Then, the corresponding value for y is $\frac{bl + a}{k}$.

Bhaskara says that the integral roots of a new equation

$$Ny^2 + \frac{l^2 - N}{k} = x^2 \qquad \dots \quad (2)$$

are given by $$y = \frac{bl + a}{k}$$

and hence $$x = \frac{al + Nb}{k}$$

where $(l^2 - N)/k$ is also an integer.

Just as we derived (2) from (1), we can derive another equation from (2) with known roots and so on, until the additive ultimately reduces to * ± 4, ± 2. or ± 1.

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* Bhaskara's own worked examples illustrate the cases where the additive reduces to -4, and -2; thus, starting with the known solution ($x = 8$, $y = 1$

As T. L Heath says, nothing is wanting to the cyclic method except the proof that it will in every case lead to the desired result, whenever N is a non-square integer; but he is wrong in supposing that Lagrange was the first to supply the proof for Bhaskara's method. Indeed, the first complete proof of Bhaskara's Cyclic Method is the one sketched in this paper.

3. The Theory underlying the Cyclic Method.

Lemma. If $a, b, a_1, b_1,$ be integers such that $ab_1 - a_1b = 1$, then the equations

$$x^2 - Ny^2 = k \qquad \dots \quad (1)$$

and

$$x'^2(a^2 - Nb^2) + 2x'y'(aa_1 - Nbb_1) + y'^2(a_1^2 - Nb_1^2) = k \dots \quad (2)$$

are so related that their integral solutions can be put into one to-one correspondence by the linear relation

$$x = ax' + a_1y'$$
$$y = bx' + b_1y'.$$

For, it is readily seen that (2) is derived from (1) by the linear substitution; solving for x', y' in terms of x, y, we get

$$x' = \frac{b_1x - a_1y}{ab_1 - a_1b} = b_1x - a_1y$$

$$y' = \frac{bx - ay}{ba_1 - ab_1} = -bx + ay.$$

This shows that, if (x', y') are integral, so are (x, y) and *vice-versa*. Thus any solution of one of the equations gives simultaneously a solution of the other.

In the language of the Theory of Numbers, the two quadratic forms on the left-side of (1) and (2) are said to be equivalent.

Now, let (a, b) be an integral solution of (1), so that

$$a^2 - Nb^2 = k.$$

Put $\qquad aa_1 - Nbb_1 = l$ and $a_1^2 - Nb_1^2 = k_1$.

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of the equation $61y^2+3=x^2$ he gets at the next stage of reduction the solution $(x=39, y=5)$ of the equation $61y^2-4=x^2$, where the additive is -4; similarly, one solution of the equation $67y^2-3=x^2$ leads successively to the solutions of the equations $67y^2+6=x^2$, $67y^2-7=x^2$, and $67y^2-2=x^2$, (the last of these equations having the additive -2).

It is readily verified that

$$\left.\begin{aligned} l^2 - kk_1 &= \mathrm{N}\,(ab_1 - a_1 b)^2 = \mathrm{N}. \\ \text{i.e.,}\quad a_1^2 - \mathrm{N}b_1^2 &= k_1 = \frac{l^2 - \mathrm{N}}{k} \end{aligned}\right\} \quad \ldots \quad (3)$$

Also, solving the two equations

$$ab_1 - a_1 b = 1$$

and

$$aa_1 - \mathrm{N}bb_1 = l \qquad \text{for } a_1,\ b_1$$

we get

$$\left.\begin{aligned} a_1 &= \frac{al + \mathrm{N}b}{k} \\ b_1 &= \frac{bl + a}{k} \end{aligned}\right\} \quad \ldots \quad (4)$$

The results (3) and (4) lay bare the clue to Bhaskara's method. In fact, they suggest a method of deriving a pair of integers (a_1, b_1) and a triplet (k, l, k_1) from a solution (a, b) of equation (1), by taking

$$b_1 = \frac{bl + a}{k},\ k_1 = \frac{l^2 - \mathrm{N}}{k},\ \text{and } \mathrm{N}b_1^2 + k_1 = a_1^2,$$

where all the letters represent integers.

Thus, if (a, b) be a pair of integral values satisfying the equation

$$\mathrm{N}y^2 + k = x^2$$

then (a_1, b_1) are also a pair of integers satisfying a new equation

$$\mathrm{N}y^2 + k_1 = x^2 \quad \ldots \quad (5)$$

Treating the roots (a_1, b_1) of (5) in the same way as we treated those of (1), we get the pair (a_2, b_2) and the triplet (k_1, l_1, k_2) such that (a_2, b_2) satisfy the equation $\mathrm{N}y^2 + k_2 = x^2$ and so on.

In the notation of the Theory of Numbers, the forms $(1, 0, -\mathrm{N})$ (k, l, k_1), (k_1, l_1, k_2), ... obtained by the above method are all equivalent.

Bhaskara's proposition is that the k's ultimately reach the values ± 4, ± 2, or ± 1.

We now proceed to show that the k's reach 1 always, but they need not necessarily pass through the values $\pm 2, \pm 4, -1$ before reaching 1.

If they do, however, take these values, Bhaskara's idea is to take a short-cut from there to the required solution. (*Vide*, Bhaskara's own solution to the equations $61y^2 + 1 = x^2$ and $67y^2 + 1 = x^2$, pp. 194—198, *Bijaganita*).

Recurrence Formulae.

The relations between the successive pairs (a, b), (a_1, b_1), ... as well as those between the successive triplets (k, l, k_1), (k_1, l_1, k_2), ... are as follows:

$$\left.\begin{array}{l} a^2 - Nb^2 = k; \\ Nb_1^2 + k_1 = a_1^2,\ a_1 = \dfrac{al + Nb}{k},\quad b_1 = \dfrac{bl + a}{k},\quad k_1 = \dfrac{l^2 - N}{k}; \\ Nb_2^2 + k_2 = a_2^2,\ a_2 = \dfrac{a_1 l_1 + Nb_1}{k_1},\ b_2 = \dfrac{b_1 l_1 + a_1}{k_1},\ k_2 = \dfrac{l_1^2 - N}{k_1}; \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ Nb_r^2 + k_r = a_r^2,\ a_r = \dfrac{a_{r-1} l_{r-1} + Nb_{r-1}}{k_{r-1}},\ b_r = \dfrac{b_{r-1} l_{r-1} + a_{r-1}}{k_{r-1}}, \\ \qquad\qquad\qquad\qquad k_r = \dfrac{l_{r-1}^2 - N}{k_{r-1}}. \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{l} \text{Now,}\quad a_2 = \dfrac{a_1 l_1 + Nb_1}{k_1} = \dfrac{a_1 l_1}{k_1} + \dfrac{Na}{kk_1} + \dfrac{l}{k_1}\left(a_1 - \dfrac{al}{k}\right) \\ \qquad = a_1 \cdot \dfrac{l + l_1}{k_1} + \dfrac{a(N - l^2)}{kk_1} \\ \qquad = a_1 \cdot \dfrac{l + l_1}{k_1} - a \\ \text{Similarly, } b_2 = b_1 \cdot \dfrac{l + l_1}{k_1} - b \end{array}\right\} \ldots \quad (7)$$

Since $ab_1 - ba_1 = a_1b_2 - b_1a_2 = \ldots\ldots = 1$, and we may take a, b to be prime to each other and have the same sign, it follows that the pairs (a_1, b_1), (a_2, b_2), ... also contain relatively prime integers having the same sign.

If $\dfrac{l + l_1}{k_1}$ be not an integer, then k_1 or some one of its factors should go into both a_1 and b_1, which is contrary to the fact that a_1 and b_1 are relatively prime. Hence, $\frac{l + l_1}{k_1}$ must be an integer.

We may write (7) in the more general form as giving the relation between any three successive pairs (a_r, b_r), (a_{r+1}, b_{r+1}), and (a_{r+2}, b_{r+2}) thus:

$$\left.\begin{aligned} a_{r+2} &= a_{r+1} \cdot \frac{l_r + l_{r+1}}{k_{r+1}} - a_r \\ b_{r+2} &= b_{r+1} \cdot \frac{l_r + l_{r+1}}{k_{r+1}} - b_r \end{aligned}\right\} \quad \ldots \quad (8)$$

where $(l_r + l_{r+1}) / k_{r+1}$ is an integer.

Incidentally, we may also notice the relation

$$a_r - b_r l_{r-1} = - b_{r-1} k_r. \quad \ldots \quad (9)$$

In the usual notation of the Theory of Numbers, the triplets

$$(k, l, k_1) \; (k_1, l_1, k_2), \; \ldots \; \ldots$$

represent *equivalent adjacent* quadratic forms with determinant N, since

$$l^2 - kk_1 = l_1^2 - k_1 k_2 = \ldots \ldots = N.$$

and $\quad l + l_1 \equiv 0 \pmod{k_1}, \; l_1 + l_2 \equiv 0 \pmod{k_2}, \; \ldots$

It is important to note that these latter congruences can easily take the place of Bhaskara's indeterminate equations

$$\frac{b_1 x + a_1}{k_1} = y, \; \frac{b_2 x + a_2}{k_2} = y, \; \ldots \ldots$$

Bhaskara's Condition for Reduction.

We will now make use of the important condition 'अल्पं शेषकं यथा' in Bhaskara's method of reduction which distinguishes it from the reduced forms due to Gauss, Lagrange and others.

Bhaskara's condition is that l_r should be such a value of x in the indeterminate equation

$$\frac{b_r x + a_r}{k_r} = y$$

that $| l_r^2 - N |$ is the least. $\quad \ldots \quad (10)$

The values of x satisfying (10) are, by virtue of (9) above, of the form $-l_{r-1} + n \mid k_r \mid$ where n can take all integral values, positive, zero, and negative. In this singly infinite system of values, there are bound to be a set of two integers (one of which is less and the other greater than $\sqrt{N}$) in the immediate neighbourhood of $+\sqrt{N}$ and similarly another in the immediate vicinity of $-\sqrt{N}$. The squares of these four * integers are evidently nearer to N than the square of any other value of x in the equation (10). According to Bhaskara we have to choose the nearest of these four squares to N.† An exceptional case may occur when the root corresponding to the nearest square leads back to the previous step in the process of reduction. In this case, the root corresponding to the nearest of the remaining squares should be chosen. This exceptional case is not explicitly noted by Bhaskara though it may be implied in his cryptic expression ‡

We will first consider the case when $\mid k_r \mid > \sqrt{N}$.

Let the least positive value of x in (10) be L.

Then $L < \mid k_r \mid$ and the four squares in question which are nearer to N than all the others are

$$L^2, \ (L + \mid k_r \mid)^2, \ (L - \mid k_r \mid)^2 \text{ and } (L - 2 \mid k_r \mid)^2$$

of which,

$$(L + \mid k_r \mid)^2 \text{ and } (L - 2 \mid k_r \mid)^2$$

are both greater than k_r^2 and therefore greater than N, and it can be verified that the nearer of these two squares is always further from N than the nearer of the other two squares, *viz.*, L^2, and $(L - \mid k_r \mid)^2$ whether the latter be both greater than N or less than N or only one of them is less than N.

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* Sometimes either three or two due, to the co-incidence of some of the integers.

† This point has been completely overlooked by writers on Bhaskara. It is interesting to note that, for Gauss's reduction, it is sufficient if we take that value of x in (10) which is nearest to $\sqrt{N}$ and less than $\sqrt{N}$. Thus Bhaskara's condition goes a step further than Gauss's and lacks thereby a certain amount of simplicity.

‡ Bhaskara seems to recognise this necessary modification as for example in his own solution of the equation $61y^2 + 1 = x^2$.

For let $(L + |k_r|)^2$ be nearer to N than $(L - 2|k_r|)^2$: and so $|k_r| - L > L$. If $(L - |k_r|)^2$ is nearer to N than L^2, we have obviously $(L + |k_r|)^2 - N > N - (L - |k_r|)^2$ when $(L - |k_r|)^2 < N$, and $(L + |k_r|)^2 - N > (L - |k_r|)^2 - N$ when $(L - |k_r|)^2 > N$.

If, however, L^2 be nearer to N than $(L - |k_r|)^2$, then, evidently $(L + |k_r|)^2 - N > L^2 - N$ when $L > \sqrt{N}$. When $L < \sqrt{N}$, $|k_r| - L$ cannot also be less than $\sqrt{N}$, for this implies

$$\sqrt{N} > |k_r| - L > L$$

which contradicts the assumption that

$$L^2 \text{ is nearer to N than } (L - |k_r|)^2.$$

Hence, $$|k_r| - L > \sqrt{N} > L$$

$$\therefore \quad (L + |k_r|)^2 - N > (|k_r| - L)^2 - N > N - L^2.$$

A similar argument applies to the case where $(L - 2|k_r|)^2$ is nearer to N than $(L + |k_r|)^2$, to prove that $(L - 2|k_r|)^2 - N$ is greater than the lesser of the two differences

$$N \sim L^2 \text{ and } N \sim (L - |k_r|)^2.$$

From the above discussion, it becomes evident that the value l_r satisfying Bhaskara's condition is either

$$L \text{ or } L - |k_r| \quad \text{when } |k_r| > \sqrt{N}.$$

Hence $$|l_r| < |k_r|,$$

Further, since $$l_r^2 - k_r k_{r+1} = N,$$

we have $$|k_r| \cdot |k_{r+1}| < l_r^2 \text{ or } N$$

and in either case $$|k_{r+1}| < |k_r|$$

Therefore, as long as the k's are greater than $\sqrt{N}$. Bhaskara's method of reduction leads to smaller and smaller numerical values for k, so that at some stage

$$|k_r| < \sqrt{N}, \text{ while } |k| > |k_1| > |k_2| > \ldots > k_{r-1} > \sqrt{N}.$$

Let us now consider the case $| k_r | < \sqrt{N}$. Since $| l_r^2 - N |$ is least, we must have

$$|(l_r \pm k_r)^2 - N| > |l_r^2 - N| \qquad \dots \quad (11)$$

But, here we take into account only the squares of the number-pairs nearest to $\sqrt{N}$ or $-\sqrt{N}$ and not the nearest of all the four squares as contemplated on p. 235.*

The inequality relation (11) can be transformed as follows :

(i) When $l_r^2 > N$, $| l_r | - | k_r | < \sqrt{N}$ and so

$$N - (| l_r | - | k_r |)^2 > l_r^2 - N$$

i.e., $$N - l_r^2 + 2 | l_r | \, | k_r | - k_r^2 > l_r^2 - N$$

i.e., $$| k_r | (2 | l_r | - | k_r |) > 2 (l_r^2 - N)$$
$$> 2 | k_r | \, | k_{r+1} |$$

$\therefore$ $$| l_r | > | k_{r+1} | + \tfrac{1}{2} | k_r | \qquad \dots \quad (12)$$

Squaring both sides,

$$l_r^2 > k_{r+1}^2 + \tfrac{1}{4} k_r^2 + | k_r | \, | k_{r+1} |$$

$\therefore$ $$k_{r+1}^2 + \tfrac{1}{4} k_r^2 < N \dagger \qquad \dots \quad (13)$$

(ii) When $l_r^2 < N$, $| l_r | + | k_r | > \sqrt{N}$ and so

$$(| l_r | + | k_r |)^2 - N > N - l_r^2$$

from which we easily derive as in (i)

$$| l_r | > | k_{r+1} | - \tfrac{1}{2} | k_r | \qquad \dots \quad (14)$$

which, on squaring, leads again to (13).

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* In this connection, it may be mentioned, that if p, q be positive integers nearest to $\sqrt{N}$ and p', q' the negative integers nearest to $-\sqrt{N}$, where p, q, p', q' are the values of x satisfying the condition (10), then the two squares nearest to N can be only one of the sets :

$$(p^2, p'^2), \ (p^2, q'^2), \ (q^2, p'^2), \ (q^2, q'^2).$$

Suppose, for example, (p^2, p'^2) are the nearest squares to N and we have to reject one of them say p^2, as it leads back to the previous step in Bhaskara's method of reduction. Then the next square we have to choose is either p'^2 or q'^2 which corresponds to the number-pair nearest to $-\sqrt{N}$, and thus even in the exceptional case the relation (11) holds.

† Bhaskara's condition of reduction (10) amounts to more than (13). It implies further that $| k_{r+1} |$ should be the least of the possible values with the exception of $| k_{r-1} |$.

Thus the condition $k_{r+1}^2 + \frac{k_r^2}{4} < N$ is equivalent to

(i) $(l_r \pm k_r)^2 \sim N > l_r^2 \sim N$

(ii) $| l_r | > \frac{1}{2} | k_r | + | k_{r+1} |$ when $l_r^2 > N$

and (iii) $| l_r | > | k_{r+1} | - \frac{1}{2} | k_r |$ when $l_r^2 < N$.

Similarly, the condition $k_r^2 + \frac{k_{r+1}^2}{4} < N$ can be seen to be equivalent to (i) $(l_r \pm k_{r+1})^2 \sim N > l_r^2 \sim N$,

and (ii) $| l_r | > \frac{1}{2} | k_{r+1} | + | k_r |$, or

$$| l_r | > | k_r | - \frac{1}{2} | k_{r+1} |$$

according as l_r^2 is greater or less than N.

Cor. (1) In every case, when $| k_r | < \sqrt{N}$, $| k_{r+1} | < \sqrt{N}$.

Cor. (2) Since $l_r^2 - k_r k_{r+1} = N$, and $| k_r . k_{r+1} | < N$.

$$l_r < \sqrt{2N}.$$

Cor. (3) $| l_r | \sim \sqrt{N} = \frac{| k_r | | k_{r+1} |}{| l_r | + \sqrt{N}} < | k_r |$ and $| k_{r+1} |$.

Cor. (4) Since $| k_r |$, $| l_r |$, $| k_{r+1} |$ can take only a limited set of integral values, the number of forms of the type (k_r, l_r, k_{r+1}) is necessarily finite.

Reduced Forms.

When the k's have become numerically less than $\sqrt{N}$, since they cannot continuously decrease numerically, we must have at some stage

$$| k_{r+1} | \geqslant | k_r | .$$

The corresponding reduced form evidently satisfies two conditions :

$$\text{(i)} \quad k_r^2 + \frac{k_{r+1}^2}{4} < N$$

$$\text{(ii)} \quad k_{r+1}^2 + \frac{k_r^2}{4} < N$$

We shall call a form (A, B, C) of determinant N. a ***reduced Bhaskara form*** (or simply a *Bhaskara form*), if it satisfies the conditions

$$A^2 + \frac{C^2}{4} < N,\; C^2 + \frac{A^2}{4} < N.$$

In the method of reduction advocated by Bhaskara, the existence of such reduced forms as (A, B, C) has now been proved.

Incidentally, we prove also the following

THEOREM. A reduced Bhaskara form can always be found equivalent to any proposed form.

Properties of Bhaskara Forms.

Let (k_r, l_r, k_{r+1}) be a Bhaskara form and $(k_{r+1}, l_{r+1}, k_{r+2})$ be an adjacent form obtained from the former by Bhaskara's method.

Since $(l_r \pm k_{r+1})^2 \sim N > l_r^2 \sim N$, (*vide* p. 238), $-l_r$ is a possible value of l_{r+1}; but this has to be rejected as it leads back to k_r, and so we have to choose for l_{r+1} one of the two numbers nearest to $+\sqrt{N}$ or $-\sqrt{N}$ according as $-l_r$ belongs to the number pair nearest to $-\sqrt{N}$ or $+\sqrt{N}$.

This shows that l_{r+1} and l_r are of the same sign. Further, it is easily seen, from the relative magnitudes of the two squares nearest to N, that

$$l_{r+1} \sim l_r < |\, k_{r+1} \,|.$$

We may suppose, for convenience, that l_r is positive. The proofs applicable to this case apply also to the case where l_r is negative, for we may write $|\, l_r \,|$ for l_r.

With these preliminaries settled, we can now prove the theorem.

THEOREM I. *A successor of a Bhaskara form is also a Bhaskara form.*

If $(k_{r+1}, l_{r+1}, k_{r+2})$ be a successor to the reduced form (k_r, l_r, k_{r+1}) then it is evident that the successor is also a reduced form in the following cases :

(i) $|k_{r+1}| \leqslant |k_{r+2}|$.

(ii) $|k_{r+2}| \leqslant |k_r|$.

(iii) $l_{r+1} > \sqrt{N}, l_{r+1} \geqslant \frac{3}{2}|k_{r+1}|$, and $|k_{r+1}| > |k_{r+2}| > |k_r|$.

(iv) $l_{r+1} < \sqrt{N}, l_{r+1} > |k_{r+1}|$, and $|k_{r+1}| > |k_{r+2}| > |k_r|$.

So, the theorem remains to be proved for the remaining cases :

(v) $l_{r+1} > \sqrt{N}, l_{r+1} < \frac{3}{2}|k_{r+1}|$, and $|k_{r+1}| > |k_{r+2}| > |k_r|$

and (vi) $l_{r+1} < \sqrt{N}, l_{r+1} < |k_{r+1}|$, and $|k_{r+1}| > |k_{r+2}| > |k_r|$

Case (v). Since $l_{r+1}^2 - N = |k_{r+1} \cdot k_{r+2}| > |k_r||k_{r+1}|$

and $$|l_r^2 - N| = |k_r k_{r+1}|,$$

we have $$l_{r+1} > l_r, \text{ whether } l_r \gtrless \sqrt{N}$$

$\therefore$ $$l_r + l_{r+1} < 2\, l_{r+1} < 3\,|k_{r+1}|$$

Also, $$l_{r+1} > \sqrt{N} > |k_{r+1}|$$

and $$l_r + l_{r+1} \equiv 0 \pmod{|k_{r+1}|}$$

Hence, $$l_r + l_{r+1} = 2|k_{r+1}|$$

But $$l_{r+1} - l_r < |k_{r+1}|$$

i e., $$2|k_{r+1}| - 2\, l_r < |k_{r+1}|$$

which implies that $$l_r < |k_{r+1}| < \sqrt{N}.$$

Now, $$l_{r+1}^2 - N > N - l_r^2$$

i.e., $$\{2|k_{r+1}| - l_r\}^2 + l_r^2 > 2\,N$$

i.e., $$l_r^2 + 2\,k_{r+1}^2 - 2\,l_r|k_{r+1}| > N$$

i.e., $$2\,k_{r+1}^2 - 2\,l_r|k_{r+1}| > N - l_r^2$$
$$> |k_r k_{r+1}|$$

i.e., $$2|k_{r+1}| - 2\,l_r > |k_r|$$

i.e., $$l_r < |k_{r+1}| - \tfrac{1}{2}|k_r|$$

which contradicts the condition

$$l_r > |k_{r+1}| - \tfrac{1}{2}|k_r|$$

equivalent to $$k_{r+1}^2 + \tfrac{1}{4}k_r^2 < N.$$

This investigation reveals that *Case* (v) is an impossible one.

Case (vi). As before, $| l_{r+1}^2 - N | > | l_r^2 - N |$.

But, since $$l_{r+1} < \sqrt{N},$$

we get $$N - l_{r+1}^2 > N - l_r^2 \text{ or } l_r^2 - N$$

and in either case $$l_r > l_{r+1}.$$

We know that $$l_r - l_{r+1} < | k_{r+1} |$$

$$\therefore \quad l_r < l_{r+1} + | k_{r+1} |$$
$$< 2 | k_{r+1} |$$

so that, $$l_r + l_{r+1} < 3 | k_{r+1} |.$$

Since $$l_r + l_{r+1} \equiv 0 \pmod{| k_{r+1} |}$$

we have $$l_r + l_{r+1} = | k_{r+1} | \text{ or } 2 | k_{r+1} |.$$

Again, $(l_{r+1})^2 + | k_{r+1} | . | k_{r+2} | = N = l_r^2 \pm | k_r | . | k_{r+1} |$

$$\therefore \quad l_r^2 - l_{r+1}^2 = | k_{r+1} | \{ | k_{r+2} | \mp | k_r | \}$$

If $l_r + l_{r+1} = | k_{r+1} |$, then $l_r - l_{r+1} = | k_{r+2} | \mp | k_r |$,

so that $l_r = \frac{1}{2} | k_{r+1} | + \frac{1}{2} | k_{r+2} | \mp \frac{1}{2} | k_r | < | k_{r+1} | \mp \frac{1}{2} | k_r |$

the upper or lower sign to be taken in each case according as l_r is less or greater than $\sqrt{N}$. This contradicts the condition that (k_r, l_r, k_{r+1}) is a reduced form (*vide* p. 238). Hence $l_r + l_{r+1} \neq | k_{r+1} |$.

Next, if $l_r + l_{r+1} = 2 | k_{r+1} |$, then $l_r - l_{r+1} = \frac{1}{2} | k_{r+2} | \mp \frac{1}{2} | k_r |$,

so that $$l_{r+1} = | k_{r+1} | - \tfrac{1}{4} \{ | k_{r+2} | \mp | k_r | \}$$
$$> | k_{r+1} | - \tfrac{1}{2} | k_{r+2} |$$

which is equivalent to the condition $k_{r+1}^2 + k_{r+2}^2 / 4 < N$.

Thus in all cases, we prove that the form $(k_{r+1}, l_{r+1}, k_{r+2})$ satisfies the conditions

$$k_{r+2}^2 + k_{r+1}^2 / 4 < N$$

and $$k_{r+1}^2 + k_{r+2}^2 / 4 < N.$$

Hence, the theorem is proved.

THEOREM II. *Two different Bhaskara forms cannot have the same successor.*

If possible, let (k_r, l_r, k_{r+1}) and $(k_{r+1}, l_{r+1}, k_{r+2})$ be two consecutive Bhaskara forms, as also (k', l', k_{r+1}) and $(k_{r+1}, l_{r+1}, k_{r+2})$

Then
$$l_r + l_{r+1} \equiv 0 \pmod{|k_{r+1}|}$$
$$l' + l_{r+1} \equiv 0 \quad (\quad ,, \quad)$$
$$\therefore \quad l' \sim l_r \equiv 0 \quad (\quad ,, \quad) \qquad \ldots (1)$$

But, since* $|l_r \sim \sqrt{N}| < |k_{r+1}|$ and $|l' \sim \sqrt{N}| < |k_{r+1}|$,
$$\therefore \quad |l' - l_r| < |l_r - \sqrt{N}| + |l' - \sqrt{N}| < 2|k_{r+1}| \ldots (2)$$
From (1) and (2), we infer
$$|l' - l_r| = 0 \text{ or } |k_{r+1}|$$

From the condition of reduction,
$$|(l' \pm k_{r+1})^2 - N| > |l'^2 - N|$$
which reduces to
$$|l^2_r - N| > |l'^2 - N|$$
where
$$|l' - l_r| = |k_{r+1}|$$
A similar argument proves also the contradictory result
$$|l'^2 - N| > |l_r^2 - N|$$
$$\therefore \quad |l' - l_r| \neq |k_{r+1}|.$$

Hence $|l' - l_r| = 0$, *i.e.* $l' = l_r$, from which it follows that $k' = k_r$; *i.e.* the two forms (k_r, l_r, k_{r+1}) and (k', l', k_{r+1}) are identical.

Thus, the theorem is proved.

THEOREM III *The Bhaskara forms repeat in a cycle and the first member of the cycle is the same as the first Bhaskara form obtained in the course of reduction.*

For, if otherwise, two different Bhaskara forms (*viz.*, the predecessor of the first member of the cycle, as well as the last member), will have the same successor.

THEOREM. IV *Two different cycles of Bhaskara forms are non-equivalent, and so all equivalent Bhaskara forms belong to the same cycle.*

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* *Vide* p 238, Cor 3,

To prove this, we have to examine the structure of a complete cycle of these forms and perceive the relation between them and Gauss-forms

Let $(k, l, k_1), (k_1, l_1, k_2), \ldots (k_s, l_s, k_{s+1}), \ldots \ldots$

be a cycle of Bhaskara forms with the l's all positive.

If $l_r < \sqrt{N}$, the form (k_r, l_r, k_{r+1}) is readily seen to be a Gauss-form satisfying the conditions $\sqrt{N} - l_r < | k_r | < \sqrt{N} + l_r$, where $N \equiv l_r^2 - k_r k_{r+1}$ (*vide* p. 74. Mathews' Theory of Numbers).

If $l_r > \sqrt{N}$, the proper unitary substitution $\begin{pmatrix} 1, & \pm 1^* \\ 0, & 1 \end{pmatrix}$ transforms (k_r, l_r, k_{r+1}) into the equivalent form $(k_r, l_r - | k_r |, k_r')$, where

$$| k_r' | = 2 l_r - | k_r | - | k_{r+1} | .$$

Since $\sqrt{N} - l_r + | k_r | < | k_r | < \sqrt{N} + l_r - | k_r |$,

the latter form is a reduced Gauss-form.

Thus, we see that with every Bhaskara form, there is always an associated equivalent Gauss-form,† the associated form being identical with the Bhaskara form when the middle co-efficient is less than $\sqrt{N}$.

Now let $(k_r, l_r, k_{r+1}), (k_s, l_s, k_{s+1})$ be two forms in Bhaskara's cycle (where l_r, l_s are each less than $\sqrt{N}$, so that the forms are also Gauss-forms) with the intervening forms having their middle co-efficients all greater than $\sqrt{N}$. We can now replace all these intervening forms by their equivalent Gauss-forms, and form the sequence:

$$(k_r, l_r, k_{r+1}), (k_{r+1}, l_{r+1} - | k_{r+1} |, k_{r+1}'), (k_{r+2}, l_{r+2} - | k_{r+2} |, k_{r+2}')$$
$$\ldots \ldots (k_{s-1}, l_{s-1} - | k_{s-1} |, k_{s-1}'), (k_s, l_s, k_{s+1})$$

in which all the forms are Gauss's.

We shall now interpolate between any two consecutive forms of the above sequence (except the first two) another Gauss-form and thus get a complete succession of unique‡ Gauss-forms from (k_r, l_r, k_{r+1}) to (k_s, l_s, k_{s+1}). For example,

between $(k_{r+1}, l_{r+1} - | k_{r+1} |, k'_{r+1})$

and $(k_{r+2}, l_{r+2} - | k_{r+2} |, k'_{r+2})$

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* The upper sign is to be taken when k_r is negative and the lower sign when k_r is positive.

† Not *vice-versa*.

‡ *Vide* p. 76, Art. 70. Mathews' Theory of Numbers.

we may put in the Gauss-form

$$(k'_{r+1},\ l_{r+1} - |k_{r+2}|,\ k_{r+2})^{*}$$

The latter form cannot be also a Bhaskara form, for,

$$|k'_{r+1}| - \tfrac{1}{2}|k_{r+2}| = 2\,l_{r+1} - |k_{r+1}| - \tfrac{3}{2}|k_{r+2}| > l_{r+1} - |k_{r+2}|$$

owing to the condition $|k_{r+1}| + \frac{1}{2}|k_{r+2}| < l_{r+1}$.

The above process enables us, in general, to convert a Bhaskara cycle into a unique Gauss cycle and *vice-versa.*

Two different cycles of Bhaskara forms † cannot evidently be transformed into the same Gauss-cycle and are therefore non-equivalent. Thus, all equivalent Bhaskara forms belong to the same period.

THEOREM V. *If L^2 be the integral square nearest to N and* $K = L^2 - N$, *then $(K, L, 1)$ where L is positive, is a Bhaskara form.*

For, let $L^2 < N$, then $N - L^2 < (L + 1)^2 - N$

$$\begin{aligned} i.e.\quad & N < L^2 + L + \tfrac{1}{2} \\ i.e.,\quad & N - L^2 - \tfrac{1}{2} < L \\ i.e.,\quad & -K - \tfrac{1}{2} < L. \\ \therefore \text{ Squaring,}\quad & K^2 + K + \tfrac{1}{4} < L^2 \\ i.e.,\quad & K^2 + \tfrac{1}{4} < L^2 - K \\ & < N \end{aligned}$$

Hence $\frac{K^2}{4} + 1 < K^2 + \frac{1}{4} < N$, and the criteria for a Bhaskara form are satisfied.

The case $L^2 > N$ can also be similarly treated.

Now, putting $a = L$, $b = -1$, $a_1 = 1$, $b_1 = 0$ in the Lemma on p. *231*, we find that $(K, L, 1)$ and $(1, 0, -N)$ are equivalent forms. But the Bhaskara forms obtained with the help of a pair of known integral roots (a, b) of the equation

$$x^2 - Ny^2 = k$$

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* To prove this to be a Gauss-form, we have only to use

$$l_{r+1} > |k_{r+2}| + \tfrac{1}{2}|k_{r+1}|$$

and cor. 3 p *238.*

† Two different Bhaskara cycles cannot have any common form; for, if one form is common, all its predecessors and successors also must be common, so that the two cycles entirely coincide.

are all equivalent to (1, 0, — N). These form a cycle, which includes as a member the particular form (K, L, 1).

Thus, we ultimately get one pair of integral roots of the equation $x^2 - Ny^2 = 1$.

From this single pair, an infinite number can be obtained by the principle of composition mentioned at the beginning of this paper. This completes the proof of Bhaskara's cyclic method.

4. Relation of the Cyclic Method to the Irregular Continued Fraction.

Just in the same way as in Gauss's Theory (*vide* Mathews, pp. 68, 78, 79), each form of a Bhaskara period, say (k, l, k_1) is transformed into the next following (k_1, l_1, k_2) by the substitution

$$\begin{pmatrix} 0, & 1 \\ -1, & -\delta_1 \end{pmatrix} \text{ where } \delta_1 = \frac{l + l_1}{k_1}.$$

If ω_1, ω_1' be the principal roots (*i.e.* roots numerically less than 1) of these forms,

$$\omega_1 = \frac{1}{-\omega_1' - \delta_1} = \frac{-1}{\delta_1 + \omega_1'}.$$

Similarly, if $\begin{pmatrix} 0, & 1 \\ -1, & -\delta_2 \end{pmatrix}$ transforms (k_1, l_1, k_2) into the next form of the period, $\qquad \omega'_1 = \dfrac{-1}{\delta_2 + \omega'_2}$

and so on.

Hence $\qquad \omega_1 = \dfrac{-1}{\delta_1 +} \dfrac{-1}{\delta_2 +} \dfrac{-1}{\delta_3 +} \cdots \cdots$

which is a recurring irregular continued fraction since the δ's recur with the forms.

Now, we are in a position to connect Bhaskara's algorithm with an irregular continued fraction.

To start with, (a, b) are the roots of $Ny^2 + k = x^2$; from this, we derive successively (a_1, b_1) the roots of $Ny^2 + k_1 = x^2$,

(a_2, b_2) ,, $Ny^2 + k_2 = x^2$,

and so on.

The corresponding quadratic forms are (*vide* p. 232. supra)

$$(k, l, k_1),\ (k_1, l_1, k_2),\ \ldots\ \ldots$$

with $\qquad \delta_1 = \dfrac{l + l_1}{k_1},\ \delta_2 = \dfrac{l_1 + l_2}{k_2},\ \ldots\ \ldots$

(δ's being integers).

The principal root of the form (k, l, k_1) is

$$\frac{-l+\sqrt{N}}{k}$$

and we get

$$\frac{-l+\sqrt{N}}{k} = \frac{-1}{\delta_1 +}\;\frac{-1}{\delta_2 +}\;\cdots\cdots$$

We may write

$$\sqrt{N} = \cfrac{a}{\cfrac{a}{\sqrt{N}} - b + b}$$

$$= \cfrac{a}{b + \cfrac{k}{bN + a\sqrt{N}}} = \cfrac{a}{b + \cfrac{1}{a_1 + a(\sqrt{N} - l)/k}}\text{*}$$

$$= \frac{a}{b +}\;\frac{1}{a_1 +}\;\frac{-a}{\delta_1 +}\;\frac{-1}{\delta_2 +}\;\cdots\cdots$$

$$= \frac{a}{b +}\;\frac{1/a}{a_1/a +}\;\frac{-1}{\delta_1 +}\;\frac{-1}{\delta_2 +}\;\cdots\cdots\dagger$$

The successive convergents to this continued fraction are

$$\frac{a}{b},\ \frac{a_1}{b_1},\ \frac{a_2}{b_2},\ \cdots\cdots\ \frac{a_r}{b_r},\ \frac{a_{r+1}}{b_{r+1}},\ \frac{a_{r+2}}{b_{r+2}},\ \cdots\cdots;$$

for

$$a_2 = a_1\delta_1 - a = a_1 . \frac{l+l_1}{k_1} - a,$$

$$b_2 = b_1\delta_1 - b = b_1 . \frac{l+l_1}{k_1} - b,$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$

$$a_{r+2} = \delta_{r+1}\, a_{r+1} - a_r = a_{r+1} . \frac{l_r + l_{r+1}}{k_{r+1}} - a_r,$$

$$b_{r+2} = \delta_{r+1}\, b_{r+1} - b_r = b_{r+1} . \frac{l_r + l_{r+1}}{k_{r+1}} - b_r,$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$

---

* Since $a_1 = \frac{al + Nb}{k}$ (*Vide* p. 233.)

† For example,

$$\sqrt{67} = \frac{8}{1 +}\;\frac{1/8}{(-41/8) +}$$

$$\left[\frac{-1}{2 +}\;\frac{(-1)}{(-2) +}\;\frac{(-1)}{(-9) +}\;\frac{(-1)}{(-2) +}\;\frac{(-1)}{2 +}\;\frac{(-1)}{(-5) +}\;\frac{(-1)}{16 +}\;\frac{(-1)}{(-5)}\right]$$

the part within the brackets being the recurring portion of the C. F.

Thus, Bhaskara's method of finding successively the roots (a_1, b_1), (a_2, b_2), ... is equivalent to finding the successive convergents of the irregular continued fraction

$$\sqrt{N} = \frac{a}{b+} \frac{1/a}{a_1/a+} \frac{-1}{\delta_1+} \frac{-1}{\delta_2+} \cdots \cdots$$

This shows clearly that Bhaskara's method is not related to the Lagrangean or Eulerian method involving the simple C. F expansion of $\sqrt{N}$, as is generally believed to be the case by T. L Heath and others.

One point, however, is worth remarking. Lagrange's chain of reductions by which he is able to prove the important theorem that any integral solution of $x^2 - Ny^2 = k$ ($|k| > \sqrt{N}$), must be deducible from the solution of one or other of a finite group of equations of the type $x^2 - Ny^2 = k'$ ($|k'| < \sqrt{N}$) is almost the same as Bhaskara's, except for a slight variation in the choice of roots of a linear indeterminate equation. Though the methods of reduction appear to be more or less the same, Bhaskara and Lagrange apply them as it were at opposite ends with opposite aims in view. While Bhaskara tries to solve $x^2 - Ny^2 = 1$ from a known solution of the equation $x^2 - Ny^2 = k$, Lagrange attempts the converse problem of solving $x^2 - Ny^2 = k$ with the known solutions of $x^2 - Ny^2 = k'$ where $|k'| < \sqrt{N}$, (*vide* Chrystal's Text-Book of Algebra, pp. 482—185).

5. Bhaskara and Fermat

Since Bhaskara clearly mentions the existence of an infinite number of solutions (पदानामानन्त्यम्), it cannot be said that Fermat was the first* to assert that the equation $x^2 - Ny^2 = 1$ (N being a non-square integer) always has an unlimited number of solutions in integers. Further, while Fermat claims that he can prove his assertion 'by the method of *descente* applied in a quite special manner,' we have presumably in Bhaskara's work a similar method, For, the cyclic method possesses the characteristics of the method of descent, starting as it does from the equation $x^2 - Ny^2 = k$ with known roots and

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* This remark is due to T L. Heath (*Vide* his *Diophantus*, p. 285)

successively reducing $|k|$, if $|k| > \sqrt{N}$, to a value less than $\sqrt{N}$, and thereafter other k's all numerically less than $\sqrt{N}$, finally ending in a cycle containing the desired *Kshepa*, unity. This anticipation of Fermat is perhaps sufficient in itself to justify Hankel's praise of the Indian method that 'it is certainly the finest thing achieved in the Theory of Numbers before Lagrange.' Remarkably enough, one of Bhaskara's worked examples, *viz.*, to find the roots of the equation $61y^2 + 1 = x^2$ is just one of those proposed by Fermat in his letter to Frénicle, of February, 1657 (Quel est, par example le plus petit quarré qui, multipliant 61, en prenant l' unité, fasse un quarré ?) * While Fermat chose this equation evidently for its difficulty, Bhaskara's reason for giving it was just the opposite, *viz*., to point out that it is one of those *favourable* cases where the solution can be made to depend upon the solution of $Ny^2 \pm 4 = x^2$. For, from the solution ($x = 8$, $y = 1$) of the equation $61y^2 + 3 = x^2$, Bhaskara's method immediately gives in the next stage of reduction the roots

$$x = 39, \; y = 5$$

for the equation

$$61y^2 - 4 = x^2.$$

Applying repeatedly the principle of composition of forms, Bhaskara obtains

$$x = 1766319049, \; y = 226153980$$

as the required roots.

Finally, it may be remarked that the early recognition by the ancient Hindus † of the importance of the fundamental equations $Ny^2 \pm 1 = x^2$ and $Ny^2 \pm 4 = x^2$ in the solution of the indeterminate equations of the second degree testifies to their instinctive grasp of the true nature of the problem.

MADRAS, 20*th* *June* 1930. A. A. KRISHNASWAMI AYYANGAR, M.A., L.T.

* *Oeuvres de Fermat* II, p. 334

† In the same way, in the case of the linear indeterminate equation, they were the first to recognise the fundamental nature of the equation $ay \pm 1 = bx$,