DEPARTMENT OF MATHEMATICS

Ma162 Solved Final EXAM Spring 2004

This exam illustrates many of the problems that are expected to be on the next final. However, you should not assume that problem types absent from this exam need not be studied.

You need to study various old homeworks and earlier exams. The official declaration of what to expect on the final is to be found on the last WHS homework 15 by clicking on the link on top.

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1. A factory manufactures plates, cups and vases. Each plate requires 4 oz. of clay, 5 minutes of shaping time and 5 minutes of painting time. Each cup requires 4 oz. of clay, 6 minutes of shaping time and 3 minutes of painting time. Each vase requires 3 oz. of clay, 6 minutes of shaping time and 7 minutes of painting time.

   The factory had 165 lbs. of clay on hand, 59 hours of capacity on the shaping machine and 46 hours worth of labor for painting. The manager wants to use up all the resources as fully as possible and remembers that he has to write down and solve equations.

   (a) Help him write down the necessary equations. Be sure to name all the variables explicitly and explain the meaning of each equation. Let p, c, v be the numbers of plates, cups and vases produced respectively. The conditions are:

$$\left[\begin{array}{l|r|c|r} \textit{Description} & LHS & \textit{sign} & RHS \\ \text{Clay used} & 4p+4c+3v & = & (165)(16) \\ \text{Shaping time} & 5p+6c+6v & = & (59)(60) \\ \text{Painting time} & 5p+3c+7v & = & (46)(60) \end{array}\right]$$

   Be sure to pay attention to the units so both sides of the equations have the same units. Usually, the $=$ signs are replaced by $\leq$ or $\geq$. In this problem, it is given that we have equations and not inequalities!

   (b) After seeing the equations, the manager recalls that if he makes a suitable augmented matrix and feeds it to his computer program, he will get an answer back and he can then use it to determine an appropriate production plan. Write down the necessary augmented matrix for him. **Do not start any reductions on the matrix!**

$$\left[\begin{array}{ccc|c} p & c & v & RHS \\ \hline 4 & 4 & 3 & 2640 \\ 5 & 6 & 6 & 3540 \\ 5 & 3 & 7 & 2760 \end{array}\right]$$

   (c) When the manager opens his file, he sees his hand calculations of an old linear equations' problem. He sees the following augmented matrix with the words "now it is easy to finish!" next to it. He decides to solve just for fun. What solution did he find?

$$\left[\begin{array}{cccc|r} x & y & z & w & \textit{constant} \\ 1 & 2 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 3 \end{array}\right]$$

This is in REF. So: *solve by backsubstitution. First* $w = -3$ *from the last equation. Then* $z = -1$ *from the third equation. The second gives* $y = 5 - z = 5 - (-1) = 6$*. The first gives* $x = 3 - 2y - 2w = 3 - (12) - (-6) = -3$.

Answer:$x = -3, y = 6, z = -1, w = -3$

(d) The manager also sees the following matrix in the margin, with the words **Done!!! Just write the answer!** next to it. He knows he is supposed to be able to just write down the complete answer immediately. Can you?

$$\left[\begin{array}{ccc|r} x & y & z & constant \\ 1 & 1 & 1 & 10 \end{array}\right]$$

This is even RREF! x is a single pivot variable and so y, z are free. Answer: $x = 10 - s - t$, $y = s$, $z = t$, where s, t are arbitrary real numbers (parameters).

2. Joe's Fine Leathers makes deluxe and regular belts. Each belt uses a square yard of leather. The deluxe belt needs two hours of labor while the regular belt needs only one hour of labor. Joe has 40 square yards of leather on hand and has 60 hours of labor available during a day. The deluxe belt sells for 28$ each the regular belt sells for 12$ each. He wishes to decide how many of each type should be made in order to maximize his profit.

   Help him by answering the following:

   (a) Set up a linear programming problem by clearly stating the variables and **all the** inequalities.
   Let x be the *number* of regular belts made and let y be the number of deluxe belts made.
   The function to maximize is

$$P = 12x + 28y.$$

   The conditions are:

$$\left[\begin{array}{l|r} \textit{Description} & \textit{Condition} \\ \text{Positivity} & x \geq 0, y \geq 0 \\ \text{Leather restriction} & x + y \leq 40 \\ \text{Work time restriction} & x + 2y \leq 60 \end{array}\right]$$

   (b) Make a suitable sketch indicating the feasible region. Be sure to list the coordinates of all the corner points. The region is the inside of a polygon with corners $A(0,0), B(40,0), C(20,20), D(0,30)$.

   (c) Use you sketch to determine the production schedule for Joe.
   The values of the function P at the four corners are respectively: $(0, 480, 800, 840)$. Hence the maximum value occurs at the point $(0, 30)$ so he makes 30 deluxe belts and no regular belts for a maximum sale of \$840.

   (d) Joe hears that he can sell the regular belt for 16$ in a certain store. Should his production plan change if he uses this new store? If so, what is the new schedule?
   With the new function $P = 16x + 28y$ his numbers at corner points become: $(0, 640, 880, 840)$, so he now must make 20 belts of each type with the new sale of \$880.

   (e) **Extra (2 pts.)** Joe starts thinking. "How much should the price of a regular belt go up before I need to change my original production schedule?" Can you decide the answer for him?

Set the new price of the regular belt as $12 + t$, so the function is $(12 + t)x + 28y$. The new values at corners are: $(0, 480 + 40t, 800 + 20t, 840)$. To change the original solution $D(0, 30)$, one of its adjacent points must rise to the same value as it! Thus either $0 = 840$ or $800 + 20t = 840$. This first is never possible and the second gives $t = 2$. So, as soon as the price of the regular belt jumps by at least \$2 to \$14 or more, his solution can change to $C(20, 20)$.

Indeed, his solution can be any point on the segment between $D(0, 30)$ and $C(20, 20)$.

For a further change in the soltuion, the price of a regular belt has to jump by 16 to \$28, so that both regular and deluxe belts have the same sale price!

If the price of the regular belt increases any further, he will only make regular belts, since now the "deluxe" belts have lost all their (price) advantage!

3. **Answer the following:**

(a) Suppose that we have a biased coin which shows up head 60% of time when tossed. This is tossed two times and you may assume that the tosses are independent. Determine the following probabilities.

- Probability that the first toss is a head.
  Answer:0.6
- Probability that the first and the second toss are both heads.
  Answer: $(0.6)(0.6) = 0.36$**.**
- Probability that at least one of the two tosses is a head.
  Answer:$(0.6)(0.4) + (0.4)(0.6) + (0.6)(0.6) = 0.84$**.**
  This is also the probability of NOT getting both tails, i.e. $1 - (0.4)(0.4)$**.**

(b) You are given the following five year data about students who start in Ma109 and go on to take one of the two calculus courses.

The numbers are classified by grades: Good (A,B,C) or Poor (D,E,W).

grade	Ma123	Ma113
Good	2457	317
Poor	641	64
Totals	3098	381

Determine the following probabilities.

- Probability that a random Ma109 student takes Ma123.
  Answer: $100\,[(3098)/(3098 + 381)] = 89.05$ percent.
- Probability that a random Ma109 student takes Ma113.
  Answer: $100\,[381/(3098 + 381)] = 10.95$ percent or $100 - 89.05$ percent.
- Probability that a random Ma109 student takes Ma123, given that he got a poor grade in Ma109.
  Answer: $100\,[641/(641 + 64)] = 90.92$ percent.
- Would you consider the event of getting a poor grade in Ma109 and taking Ma123 as independent? Explain your reasoning!
  Answer:
  Let E be the event of getting a poor grade in Ma109 and F the event of taking the calculus course Ma123. Then $P(E \cap F) = (641)/(3098 + 381) = 0.1842$. Also $P(E) = (641 + 64)/(3098 + 381) = 0.2026$ and $P(F) = (3098)/(3098 + 381) = 0.8905$.

Thus $P(E) \cdot P(F) = 0.1804$. This differs from $P(E \cap F)$ by 0.0038 which is clearly a small number. So we consider the two numbers as essentially equal and the two events as independent by definition!
There is another point raised in class, which may not be relevant here, but worth mentioning. Which definition of independence is more appropriate?

$$P(E \cap F) = P(E) \cdot P(F) \text{ or } P(E|F) = P(E) \text{ and } P(F|E) = P(F).$$

In our calculations (using set counts) $P(E|F) = P(E \cap F)/P(F)$ and the condition $P(E|F) = P(E)$ reduces to dividing the first equality condition above by $P(F)$ on both sides. Now, for a true equality, this would not change anything, but for an approximate equality, this can change the accuracy substantially. In particular, the first equality will turn out to be too crude a measure of independence. This, however, is the agreed condition in our WHS homework. In a concrete example, you should investigate the numbers yourself and make your own reasoning.
In our current situation, $P(F|E) = \dfrac{100(641)}{(641+64)} = 90.92$ percent and $P(F) = 89.05$ percent. Should these be called approximately the same probabilities? On the other hand, if we compare $P(E|F)$ with $P(E)$ we get the numbers $20.69, 20.26$ percent and these should be considered pretty close. Thus, the second pair of conditions does not seem to be sufficiently symmetric!
Of course, Statisticians may have other questions about the data gathering before passing this judgement and you may learn them in some future course!

4. **Answer the following:**

   (a) An urn contains 9 white balls and one black ball. We reach in and bring out three balls at once. Determine the following probabilities.

   - Probability that all three are white.
     Answer: The sample space is the choice of 3 balls from 10 or $C(10,3)$. To get three white balls, we simply choose all three balls from the 9 white balls; this number is $C(9,3)$. The answer is
     $$\frac{(9!)(3!7!)}{(3!6!)(10!)} = \frac{9 \cdot 8 \cdot 7}{10 \cdot 9 \cdot 8} = 7/10.$$
   - Probability that there is a black ball among the three.
     Answer: This is done by choosing the black ball and two of the 9 white ones. Thus the answer is $C(9,2)/C(10,3)$ which is equal to
     $$\frac{(9!)(3!7!)}{(2!7!)(10!)} = \frac{9 \cdot 8 \cdot 3 \cdot 2}{2 \cdot 10 \cdot 9 \cdot 8} = 3/10.$$
     Of course, we could have guessed this more easily as $1 - 7/10$.

   (b) You roll three fair dice and note the numbers which show up on top. Determine the following probabilities.

   - Probability of a triple, i.e. getting the same top number on all three.
     Answer: The sample space is the set of all 3-tuple of numbers between 1 and 6, so has $6^3 = 216$ elements.
     A triple is a choice (x, x, x) and there are exactly 6 distinct triples, so the probability is $6/216 = 1/36$.

- Probability of a pair of sixes, i.e. getting at least two of the top numbers as 6.
  Answer: The choices are of the type $(6, x, 6)$ or $(6, 6, x)$ or $(x, 6, 6)$. If we choose x to be between 1 and 5 then we get a total of 15 different cases. If we take $x = 6$ then we get a single case. Thus we have a total of 16 cases from among 216. The probability is 16/216.
- Probability that we get a triple or a pair of sixes. **Answer:** This can be done by a direct counting, but we illustrate the use of the formula.
  Let E be the event of getting a triple and F the event of getting a pair of sixes. Then $E \cap F$ is the event of getting the unique triple which has a pair of sixes, namely $(6, 6, 6)$. Thus the desired probability

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = 6/216 + 16/216 - 1/216 = 21/216.$$

5. **Answer the following:**

(a) You have an appointment in 30 minutes, but your friend has borrowed your car. There is a 50% chance that he comes back soon. If he does show up soon, you figure a 90% chance that you can make the appointment.

If he does not show up soon, then you need to look for a bus and then there is only a 60% chance that you can still keep the appointment. Determine the following probabilities.

- Probability that you keep your appointment by driving your own car.
  Answer: This means the friend comes back (probability 0.5) and you make the appointment with your car (probabiity 0.9), so $(0.5)(0.9) = 0.45$.
- Probability that you keep your appointment by riding a bus. **Answer:** This means that the friend does not come back (probability 0.5) and you make the appointment with a bus (probability only 0.6), so $(0.5)(0.6) = 0.30$.
- Probability that you keep your appointment (one way or the other). **Answer:** There are the above two mutually exclusive ways of doing this, so probability is $0.45 + 0.30 = 0.75$.

(b) What is the probability that at least two of five randomly chosen students in a class have a common birth-month (birth-day in the same month)? Explain your reasoning.

There are 12 months and our sample space is a set of 5-tuples of numbers $1, \cdots, 12$. This has size 12^5. The event when no two students have a comon birth-month is computed as the product $(12)(11)(10)(9)(8)$ by reasoning like this: The first student has one of the 12 birth-months, the second has one of the remaining 11, the third has one of the remaining 10 and so on. Thus probability of all distinct birth-months is

$$\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{12^5} = 0.3819.$$

Our event is the complement, so has the probability $1 - 0.3819 = 0.6181$.